ON FAREY SERIES AND DEDEKIND SUMS

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1. INTRODUCTION

As usual, the Farey series $\mathcal{F}_n$ of order n is the ascending series of irreducible fractions between 0 and 1 whose denominators do not exceed n. Thus, h/k belongs to $\mathcal{F}_n = \{\rho_0, \rho_1, \rho_2, \ldots, \rho_m\}$, where $m = \phi(1)+\phi(2)+\cdots+\phi(n)$, if $0 \le h \le k \le n$, $(h, k) = 1$; the numbers 0 and 1 are included in the forms $\frac{0}{1}$ and $\frac{1}{1}$. For example, $\mathcal{F}_5$ is:

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

The many characteristic properties of $\mathcal{F}_n$ can be found in references [1] and [3]. In this paper, we shall study the distribution problems of Dedekind sums for Farey fractions, and obtain some interesting identities. For convenience, we first introduce the definition of the Dedekind sum $S(h, q)$. For a positive integer q and an arbitrary integer h, we define

$$S(h, q) = \sum_{a=1}^{q} \left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right),$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2} & \text{if } x \text{ is not an integer;} \\ 0 & \text{if } x \text{ is an integer.} \end{cases}$$

The various arithmetical properties of $S(h, k)$ were investigated by many authors. Perhaps the most famous property of $S(h, k)$ is the reciprocity formula (see [2], [4], and [6]):

$$S(h, q) + S(q, h) = \frac{h^2+q^2+1}{12hq} - \frac{1}{4} \tag{1}$$

for all $(h, q) = 1$, $h > 0$, $q > 0$. Regarding Dedekind sums and uniform distribution, G. Myerson [5], Z. Zheng [10], and I. Vardi [7] have also obtained some meaningful results. But for any fraction a_i / b_i belonging to the Farey series $\mathcal{F}_q$, the authors are not aware of the study of the properties of $S(a_i, b_i)$. The main purpose of this paper is to study the properties of $S(a_i, b_i)$ for a_i / b_i belonging to the Farey series $\mathcal{F}_q$, and give an interesting identity. That is, we shall prove the following two main theorems.

Theorem 1: Let $0 < a \le q$ be a positive integer with $(a, q) = 1$. Then we have the identity

$$S(a, q) = \frac{1}{12}\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k} + \frac{b_k}{b_{k-1}}\right) + \frac{a}{12q} - \frac{n}{4},$$

where n is the position of $\rho_n = a/q = a_n/b_n$ in the Farey series $\mathcal{F}_q$, b_i $(0 \le i \le n)$ is the denominator of $\rho_i = \frac{a_i}{b_i}$ with $\frac{a_i}{b_i} \le \rho_n = \frac{a}{q}$ in the Farey series $\mathcal{F}_q$.

Theorem 2: Let p be a prime and let a be a positive integer with $a<p$, then we have the identity

$$\sum_{\chi(-1)=-1} \chi(a)|L(1,\chi)|^2 = \frac{\pi^2(p-1)}{12p}\left[\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k}+\frac{b_k}{b_{k-1}}\right)+\frac{a}{p}-3n\right],$$

where χ is the Dirichlet character mod p and $L(1,\chi)$ is the Dirichlet L-function corresponding character χ.

For $a=2$ and 3, from Theorem 2 and the properties of character, we immediately obtain the following two corollaries.

Corollary 1: Let p be a prime and χ be the Dirichlet odd character mod p. Then we have

$$\sum_{\chi(-1)=-1} \chi(2)|L(1,\chi)|^2 = \frac{\pi^2(p-1)^2(p-5)}{24p^2}.$$

Corollary 2: Let p be a prime and χ be the Dirichlet character modulo p. Then

$$\sum_{\chi(-1)=-1} \chi(3)|L(1,\chi)|^2 = \begin{cases} \frac{\pi^2}{36}\cdot\frac{(p-1)^2(p-10)}{p^2} & \text{if } p\equiv 1 \bmod 3; \\ \frac{\pi^2}{36}\cdot\frac{(p-1)(p-2)(p-5)}{p^2} & \text{if } p\equiv 2 \bmod 3. \end{cases}$$

It is clear that these two corollaries are an extension of Walum [8].

2. SOME LEMMAS

To complete the proof of Theorems 1 and 2, we need the following two lemmas.

Lemma 1: If h/k and h'/k' are two successive terms in $\mathcal{F}_n$, then $kh'-hk'=1$.

Proof: See Theorem 5.5 of [1].

Lemma 2: Let k and h be integers with $k\ge 3$ and $(h,k)=1$. Then we have

$$S(h,k) = \frac{1}{\pi^2 k}\sum_{d|k}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}} \chi(h)|L(1,\chi)|^2$$

where $\phi(k)$ is Euler's function.

Proof: See [9].

3. PROOF OF THE THEOREMS

In this section, we shall complete the proof of the theorems. First, we prove Theorem 1. We write the Farey fractions $\mathcal{F}_q$ as follows:

$$\frac{0}{1}, \frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}, \ldots, \frac{1}{1},$$

and suppose $\frac{a_n}{b_n}=\frac{a}{q}$.

For the successive terms $\frac{a_n}{b_n}$ and $\frac{a_{n-1}}{b_{n-1}}$, from Lemma 1 we know that

$$a_n b_{n-1} - b_n a_{n-1} = 1. \tag{2}$$

Using the properties of Dedekind sums and (2), we get

$$\begin{aligned} S(a_n, b_n) &= S(a_n b_{n-1} \overline{b_{n-1}}, b_n) \\ &= S(\overline{b_{n-1}}(1 + a_{n-1} b_n), b_n) \\ &= S(\overline{b_{n-1}}, b_n) = S(b_{n-1}, b_n). \end{aligned} \tag{3}$$

Similarly, we also have

$$\begin{aligned} S(a_{n-1}, b_{n-1}) &= S(a_{n-1} b_n \overline{b_n}, b_{n-1}) \\ &= S((a_n b_{n-1} - 1)\, \overline{b_n}, b_{n-1}) \\ &= S(-\overline{b_n}, b_{n-1}) = -S(b_n, b_{n-1}), \end{aligned} \tag{4}$$

where $\overline{b_n}$ denotes the solution x of the congruence equation $xb_n \equiv 1 \pmod{b_{n-1}}$.

So, from (3), (4), and the reciprocity formula (1), we obtain

$$S(a_n, b_n) - S(a_{n-1}, b_{n-1}) = S(b_{n-1}, b_n) + S(b_n, b_{n-1}) = \frac{b_n^2 + b_{n-1}^2 + 1}{12 b_n b_{n-1}} - \frac{1}{4}. \tag{5}$$

Hence, by expression (5) and Lemma 1, we obtain

$$\begin{aligned} S(a_n, b_n) &= S(a_{n-1}, b_{n-1}) + \frac{1}{12}\left(\frac{b_{n-1}}{b_n} + \frac{b_n}{b_{n-1}}\right) + \frac{1}{12 b_n b_{n-1}} - \frac{1}{4} \\ &= S(a_{n-1}, b_{n-1}) + \frac{1}{12}\left(\frac{b_{n-1}}{b_n} + \frac{b_n}{b_{n-1}}\right) + \frac{1}{12}\left(\frac{a_n}{b_n} - \frac{a_{n-1}}{b_{n-1}}\right) - \frac{1}{4} \\ &\cdots \\ &= \frac{1}{12}\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k} + \frac{b_k}{b_{k-1}}\right) + \frac{a}{12q} - \frac{n}{4}. \end{aligned} \tag{6}$$

From (6) and the fact that $a_n / b_n = a / q$, we immediately have

$$S(a, q) = \frac{1}{12}\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k} + \frac{b_k}{b_{k-1}}\right) + \frac{a}{12q} - \frac{n}{4}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2: Using Lemma 2, we have

$$\sum_{\substack{\chi \bmod p \\ \chi(-1) = -1}} \chi(a) |L(1, \chi)|^2 = \frac{\pi^2 (p-1)}{p} S(a, p). \tag{7}$$

Then from Theorem 1 and (7), we can easily obtain

$$\begin{aligned} \sum_{\substack{\chi \bmod p \\ \chi(-1) = -1}} \chi(a) |L(1, \chi)|^2 &= \frac{\pi^2 (p-1)}{p}\left[\frac{1}{12}\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k} + \frac{b_k}{b_{k-1}}\right) + \frac{a}{12p} - \frac{n}{4}\right] \\ &= \frac{\pi^2 (p-1)}{12p}\left[\sum_{k=1}^{n}\left(\frac{b_{k-1}}{b_k} + \frac{b_k}{b_{k-1}}\right) + \frac{a}{p} - 3n\right]. \end{aligned}$$

This completes the proof of Theorem 2.

Proof of the Corollaries: If $a = 2$, then the position of $2/p$ in the Farey fractions $\mathcal{F}_p$ is $\frac{p+3}{2}$, so $n = \frac{p+3}{2}$. Thus, from Theorem 2, we have

$$\begin{aligned}\sum_{k=1}^{\frac{p+3}{2}}\left(\frac{b_{k-1}}{b_k}+\frac{b_k}{b_{k-1}}\right) &= \frac{1}{p}+\frac{p}{p-1}+\cdots+\frac{p-\frac{p-1}{2}+1}{p-\frac{p-1}{2}}+\frac{p-\frac{p-1}{2}}{p} \\ &\quad +p+\frac{p-1}{p}+\frac{p-2}{p-1}+\cdots+\frac{p-\frac{p-1}{2}}{p-\frac{p-1}{2}+1}+\frac{p}{p-\frac{p-1}{2}} \qquad (8) \\ &= p+1+2\cdot\frac{p-3}{4}+\frac{2p-\frac{p-1}{2}+1}{p-\frac{p-1}{2}}+\frac{p-\frac{p-1}{2}}{p}.\end{aligned}$$

So, from (8) and Theorem 2, we have

$$\begin{aligned}\sum_{\chi(-1)=-1}\chi(2)|L(1,\chi)|^2 &= \frac{\pi^2(p-1)}{12p}\left[\sum_{k=1}^{\frac{p+3}{2}}\left(\frac{b_{k-1}}{b_k}+\frac{b_k}{b_{k-1}}\right)+\frac{2}{p}-\frac{3(p+3)}{2}\right] \\ &= \frac{\pi^2(p-1)}{12p}\left[p+1+2\cdot\frac{p-3}{4}+\frac{2p-\frac{p-1}{2}+1}{p-\frac{p-1}{2}}+\frac{p-\frac{p-1}{2}}{p}+\frac{2}{p}-\frac{3(p+3)}{2}\right] \\ &= \frac{\pi^2(p-1)^2(p-5)}{24p^2}.\end{aligned}$$

This proves Corollary 1.

Using Theorem 2, or the reciprocity formula (1) and Lemma 2, we may immediately deduce

$$\sum_{\chi(-1)=-1}\chi(3)|L(1,\chi)|^2 = \begin{cases}\frac{\pi^2}{36}\cdot\frac{(p-1)^2(p-10)}{p^2} & \text{if } p\equiv 1 \bmod 3; \\ \frac{\pi^2}{36}\cdot\frac{(p-1)(p-2)(p-5)}{p^2} & \text{if } p\equiv 2 \bmod 3.\end{cases}$$

This completes the proof of Corollary 2.

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