FACTORIZATIONS AND EIGENVALUES OF FIBONACCI AND SYMMETRIC FIBONACCI MATRICES

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1. INTRODUCTION

Matrix methods are a major tool in solving many problems stemming from linear recurrence relations. A matrix version of a linear recurrence relation on the Fibonacci sequence is well known as

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}.$$

We let

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & F_1 \\ F_1 & F_2 \end{bmatrix},$$

then we can easily establish the following interesting property of Q by mathematical induction.

$$Q^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}.$$

From the equation $Q^{n+1}Q^n = Q^{2n+1}$, we get

$$\begin{bmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{bmatrix} \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{2n+2} & F_{2n+1} \\ F_{2n+1} & F_{2n} \end{bmatrix},$$

which, upon tracing through the multiplication, yields an identity for each Fibonacci number on the right-hand side. For example, we have the elegant formula,

$$F_{n+1}^2 + F_n^2 = F_{2n+1}. \tag{1}$$

The sum of the squares of the first n Fibonacci numbers is almost as famous as the formula for the sum of the first n terms:

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}. \tag{2}$$

In particular, in [1], the authors gave several basic Fibonacci identities. For example,

$$F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{n-1}F_n = \frac{F_{2n-1} + F_nF_{n-1} - 1}{2}. \tag{3}$$

Now, we define a new matrix. The $n \times n$ *Fibonacci matrix* $\mathcal{F}_n = [f_{ij}]$ is defined as

$$\mathcal{F}_n = [f_{ij}] = \begin{cases} F_{i-j+1}, & i-j+1 \geq 0, \\ 0, & i-j+1 < 0. \end{cases}$$

For example,

$$\mathcal{F}_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 \\ 5 & 3 & 2 & 1 & 1 \end{bmatrix},$$

and the first column of $\mathcal{F}_5$ is the vector $(1,1,2,3,5)^T$. Thus, several interesting facts can be found from the matrix $\mathcal{F}_n$.

The set of all n-square matrices is denoted by M_n. Any matrix $B \in M_n$ of the form $B = A^*A$, $A \in M_n$, may be written as $B = LL^*$, where $L \in M_n$ is a lower triangular matrix with nonnegative diagonal entries. This factorization is unique if A is nonsingular. This is called the *Cholesky factorization* of B. In particular, a matrix B is positive definite if and only if there exists a nonsingular lower triangular matrix $L \in M_n$ with positive diagonal entries such that $B = LL^*$. If B is a real matrix, L may be taken to be real.

A matrix $A \in M_n$ of the form

$$A = \begin{bmatrix} A_{11} & 0 & & \\ 0 & A_{22} & & 0 \\ & & \ddots & \\ & 0 & & A_{kk} \end{bmatrix}$$

in which $A_{ii} \in M_{n_i}$, $i = 1, 2, \dots, k$, and $\sum_{i=1}^{k} n_i = n$, is called *block diagonal*. Notationally, such a matrix is often indicated as $A = A_{11} \oplus A_{22} \oplus \cdots \oplus A_{kk}$ or, more briefly, $\oplus \sum_{i=1}^{k} A_{ii}$; this is called the *direct sum* of the matrices $A_{11}, \dots, A_{kk}$.

2. FACTORIZATIONS

In [2], the authors gave the Cholesky factorization of the Pascal matrix. In this section we consider the construction and factorization of our Fibonacci matrix of order n by using the $(0,1)$-matrix, where a matrix is said to be a $(0,1)$-matrix if each of its entries is either 0 or 1.

Let I_n be the identity matrix of order n. Further, we define the $n \times n$ matrices S_n, $\overline{\mathcal{F}}_n$, and G_k by

$$S_0 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad S_{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

and $S_k = S_0 \oplus I_k$, $k = 1, 2, \dots$, $\overline{\mathcal{F}}_n = [1] \oplus \mathcal{F}_{n-1}$, $G_1 = I_n$, $G_2 = I_{n-3} \oplus S_{-1}$, and, for $k \geq 3$, $G_k = I_{n-k} \oplus S_{k-3}$. Then we have the following lemma.

Lemma 2.1: $\overline{\mathcal{F}}_k S_{k-3} = \mathcal{F}_k$, $k \geq 3$.

Proof: For $k = 3$, we have $\overline{\mathcal{F}}_3 S_0 = \mathcal{F}_3$. Let $k > 3$. From the definition of the matrix product and the familiar Fibonacci sequence, the conclusion follows. □

From the definition of G_k, we know that $G_n = S_{n-3}$, $G_1 = I_n$, and $I_{n-3} \oplus S_{-1}$. The following theorem is an immediate consequence of Lemma 2.1.

Theorem 2.2: The Fibonacci matrix $\mathcal{F}_n$ can be factored by the G_k's as follows: $\mathcal{F}_n = G_1 G_2 \cdots G_n$. For example,

$$\mathcal{F}_5 = G_1G_2G_3G_4G_5 = I_5(I_2 \oplus S_{-1})(I_2 \oplus S_0)([1] \oplus S_1)S_2$$

$$= \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&1&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&1&1&0\\ 0&0&1&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&1&1&0&0\\ 0&1&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 1&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix}$$

$$= \begin{bmatrix} 1&0&0&0&0\\ 1&1&0&0&0\\ 2&1&1&0&0\\ 3&2&1&1&0\\ 5&3&2&1&1 \end{bmatrix}.$$

Now we consider another factorization of $\mathcal{F}_n$. The $n \times n$ matrix $C_n = [c_{ij}]$ is defined as

$$c_{ij} = \begin{cases} F_i, & j = 1, \\ 1, & i = j, \\ 0, & \text{otherwise}, \end{cases} \qquad \text{i.e., } C_n = \begin{bmatrix} F_1 & 0 & \cdots & 0 \\ F_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_n & 0 & \cdots & 1 \end{bmatrix}.$$

The next theorem follows by a simple calculation.

Theorem 2.3: For $n \ge 2$, $\mathcal{F}_n = C_n(I_1 \oplus C_{n-1})(I_2 \oplus C_{n-2}) \cdots (I_{n-2} \oplus C_2)$.

Also, we can easily find the inverse of the Fibonacci matrix $\mathcal{F}_n$. We know that

$$S_0^{-1} = \begin{bmatrix} 1&0&0\\ -1&1&0\\ -1&0&1 \end{bmatrix}, \quad S_{-1}^{-1} = \begin{bmatrix} 1&0&0\\ 0&1&0\\ 0&-1&1 \end{bmatrix}, \quad \text{and} \quad S_k^{-1} = S_0^{-1} \oplus I_k.$$

Define $H_k = G_k^{-1}$. Then

$$H_1 = G_1^{-1} = I_n, \quad H_2 = G_2^{-1} = I_{n-3} \oplus S_{-1}^{-1} = I_{n-2} \oplus \begin{bmatrix} 1&0\\ -1&1 \end{bmatrix}, \quad \text{and} \quad H_n = S_{n-3}^{-1}.$$

Also, we know that

$$C_n^{-1} = \begin{bmatrix} F_1 & 0 & \cdots & 0 \\ -F_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -F_n & 0 & \cdots & 1 \end{bmatrix} \quad \text{and} \quad (I_k \oplus C_{n-k})^{-1} = I_k \oplus C_{n-k}^{-1}.$$

So the following corollary holds.

Corollary 2.4: $\mathcal{F}_n^{-1} = G_n^{-1}G_{n-1}^{-1} \cdots G_2^{-1}G_1^{-1} = H_nH_{n-1} \cdots H_2H_1 = (I_{n-2} \oplus C_2)^{-1} \cdots (I_1 \oplus C_{n-1})^{-1}C_n^{-1}$.

From Corollary 2.4, we have

$$\mathcal{F}_n^{-1} = \begin{bmatrix} 1&0&0&0&\cdots&0\\ -1&1&0&0&\cdots&0\\ -1&-1&1&0&\cdots&0\\ 0&-1&-1&1&\cdots&0\\ \vdots&\vdots&\ddots&\ddots&\ddots&\vdots\\ 0&\cdots&0&-1&-1&1 \end{bmatrix}. \tag{4}$$

Now we define a *symmetric Fibonacci matrix* $\mathcal{Q}_n = [q_{ij}]$ as, for $i, j = 1, 2, \dots, n$,

$$q_{ij} = q_{ji} = \begin{cases} \sum_{k=1}^{i} F_k^2, & i = j, \\ q_{i,\, j-2} + q_{i,\, j-1}, & i+1 \le j, \end{cases}$$

where $q_{1,0} = 0$. Then we have $q_{1j} = q_{j1} = F_j$ and $q_{2j} = q_{j2} = F_{j+1}$. For example,

$$\mathcal{Q}_{10} = \begin{bmatrix} 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 \\ 1 & 2 & 3 & 5 & 8 & 13 & 21 & 34 & 55 & 89 \\ 2 & 3 & 6 & 9 & 15 & 24 & 39 & 63 & 102 & 165 \\ 3 & 5 & 9 & 15 & 24 & 39 & 63 & 102 & 165 & 267 \\ 5 & 8 & 15 & 24 & 40 & 64 & 104 & 168 & 272 & 440 \\ 8 & 13 & 24 & 39 & 64 & 104 & 168 & 272 & 440 & 712 \\ 13 & 21 & 39 & 63 & 104 & 168 & 273 & 441 & 714 & 1155 \\ 21 & 34 & 63 & 102 & 168 & 272 & 441 & 714 & 1155 & 1869 \\ 34 & 55 & 102 & 165 & 272 & 440 & 714 & 1155 & 1879 & 3025 \\ 55 & 89 & 165 & 267 & 440 & 712 & 1155 & 1869 & 3025 & 4895 \end{bmatrix}.$$

From the definition of $\mathcal{Q}_n$, we derive the following lemma.

Lemma 2.5: For $j \ge 3$, $q_{3j} = F_4(F_{j-3} + F_{j-2}F_3)$.

Proof: We know that $q_{3,3} = F_1^2 + F_2^2 + F_3^2 = F_3F_4$; hence, $q_{3,3} = F_4F_3 = F_4(F_0 + F_1F_3)$ for $F_0 = 0$. By induction, $q_{3j} = F_4(F_{j-3} + F_{j-2}F_3)$. □

We know that $q_{3,1} = q_{1,3} = F_3$ and $q_{3,2} = q_{2,3} = F_4$. Also we see that $q_{4,1} = q_{1,4}$, $q_{4,2} = q_{2,4}$, and $q_{4,3} = q_{3,4}$. By induction, we have the following lemma.

Lemma 2.6: For $j \ge 4$, $q_{4j} = F_4(F_{j-4} + F_{j-4}F_3 + F_{j-3}F_5)$.

From Lemmas 2.5 and 2.6, we know $q_{5,1}$, $q_{5,2}$, $q_{5,3}$, and $q_{5,4}$. From these facts and the definition of $\mathcal{Q}_n$, we have the following lemma.

Lemma 2.7: For $j \ge 5$, $q_{5j} = F_{j-5}F_4(1 + F_3 + F_5) + F_{j-4}F_5F_6$.

Proof: Since $q_{5,5} = F_5F_6$ we have, by induction, $q_{5j} = F_{j-5}F_4(1 + F_3 + F_5) + F_{j-4}F_5F_6$. □

From the definition of $\mathcal{Q}_n$ together with Lemmas 2.5, 2.6, and 2.7, we have the following lemma by induction on i.

Lemma 2.8: For $j \ge i \ge 6$,

$$q_{ij} = F_{j-i}F_4(1 + F_3 + F_5) + F_{j-i}F_5F_6 + F_{j-i}F_6F_7 + \cdots + F_{j-i}F_{i-1}F_i + F_{j-i+1}F_iF_{i+1}.$$

Now we have the following theorem.

Theorem 2.9: For $n \ge 1$ a positive integer, $H_nH_{n-1} \cdots H_2H_1\mathcal{Q}_n = \mathcal{F}_n^T$ and the Cholesky factorization of $\mathcal{Q}_n$ is given by $\mathcal{Q}_n = \mathcal{F}_n\mathcal{F}_n^T$.

Proof: By Corollary 2.4, $H_nH_{n-1} \cdots H_2H_1 = \mathcal{F}_n^{-1}$. So, if we have $\mathcal{F}_n^{-1}\mathcal{Q}_n = \mathcal{F}_n^T$, then the theorem holds.

Let $X = [x_{ij}] = \mathcal{F}_n^{-1}\mathcal{Q}_n$. Then, by (4), we have the following:

$$x_{ij}=\begin{cases}F_j, & \text{if } i=1,\\ F_{j-1}, & \text{if } i=2,\\ -q_{i-2,j}-q_{i-1,j}+q_{ij} & \text{otherwise.}\end{cases}$$

Now we consider the case $i\geq 3$. Since $\mathcal{Q}_n$ is a symmetric matrix, $-q_{i-2,j}-q_{i-1,j}+q_{ij}=-q_{j,i-2}-q_{j,i-1}+q_{ji}$. Hence, by the definition of $\mathcal{Q}_n$, $x_{ij}=0$ for $j+1\leq i$. So, we will prove that $-q_{i-2,j}-q_{i-1,j}+q_{ij}=F_{j-i+1}$ for $j\geq i$.

In the case in which $i\leq 5$, we have $x_{ij}=F_{j-i+1}$ by Lemmas 2.5, 2.4, and 2.7.

Now suppose that $j\geq i\geq 6$. Then, by Lemma 2.8, we have

$$\begin{aligned}x_{ij}&=-q_{i-2,j}-q_{i-1,j}+q_{ij}\\&=(F_{j-i}-F_{j-i+1}-F_{j-i+2})F_4(1+F_3+F_5)+(F_{j-i}-F_{j-i+1}-F_{j-i+2})F_5F_6\\&\quad+\cdots+(F_{j-i}-F_{j-i+1}-F_{j-i+2})F_{i-3}F_{i-2}+(F_{j-i}-F_{j-i+1}-F_{j-i+3})F_{i-2}F_{i-1}\\&\quad+(F_{j-i}-F_{j-i+2})F_{i-1}F_i+F_{j-i+1}F_iF_{i+1}.\end{aligned}$$

Since $F_{j-i}-F_{j-i+1}-F_{j-i+2}=-2F_{j-i+1}$, $F_{j-i}-F_{j-i+1}-F_{j-i+3}=-3F_{j-i+1}$, and $F_{j-i}-F_{j-i+2}=-F_{j-i+1}$, we have

$$x_{ij}=F_{j-i+1}[-2F_4-2(F_3F_4+F_4F_5+\cdots+F_{i-2}F_{i-1})-F_{i-2}F_{i-1}-F_{i-1}F_i+F_iF_{i+1}].$$

Since $F_4=3$, using (3) we have

$$x_{ij}=\left[-6-2\left(\frac{F_{2(i-1)-1}+F_{i-1}F_{(i-1)-1}-1}{2}-F_1F_2-F_2F_3\right)-F_{i-2}F_{i-1}-F_{i-1}F_i+F_iF_{i+1}\right]F_{j-i+1}.$$

Since $F_{i+1}=F_i+F_{i-1}$ and by (1) we have

$$\begin{aligned}x_{ij}&=(1-2F_{i-1}F_{i-2}-F_{2i-3}-F_{i-1}F_i+F_iF_{i+1})F_{j-i+1}\\&=(1-2F_{i-1}F_{i-2}-F_{2i-3}+F_i^2)F_{j-i+1}\\&=(1-F_{i-1}^2-F_{i-2}^2-2F_{i-1}F_{i-2}+F_i^2)F_{j-i+1}\\&=(1-(F_{i-1}+F_{i-1})^2+F_i^2)F_{j-i+1}\\&=(1-F_i^2+F_i^2)F_{j-i+1}=F_{j-i+1}.\end{aligned}$$

Therefore, $\mathcal{F}_n^{-1}\mathcal{Q}_n=\mathcal{F}_n^T$, i.e., the Cholesky factorization of $\mathcal{Q}_n$ is given by $\mathcal{Q}_n=\mathcal{F}_n\mathcal{F}_n^T$. □

In particular, since $\mathcal{Q}_n^{-1}=(\mathcal{F}_n^T)^{-1}\mathcal{F}_n^{-1}=(\mathcal{F}_n^{-1})^T\mathcal{F}_n^{-1}$, we have

$$\mathcal{Q}_n^{-1}=\begin{bmatrix}3&0&-1&0&\cdots&&&0\\0&3&0&-1&\cdots&&&0\\-1&0&3&0&\cdots&&&0\\0&-1&0&3&\ddots&&&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\ddots&&\vdots\\0&0&0&0&\cdots&3&0&-1\\0&0&0&0&\cdots&0&2&-1\\0&0&0&0&\cdots&-1&-1&1\end{bmatrix}. \tag{5}$$

From Theorem 2.9, we have the following corollary.

Corollary 2.10: If k is an odd number, then

$$F_nF_{n-k}+\cdots+F_{k+1}F_1=\begin{cases}F_nF_{n-(k-1)}-F_k & \text{if } n \text{ is odd,}\\ F_nF_{n-(k-1)} & \text{if } n \text{ is even.}\end{cases}$$

If k is an even number, then

$$F_nF_{n-k}+\cdots+F_{k+1}F_1=\begin{cases}F_nF_{n-(k-1)} & \text{if } n \text{ is odd,}\\ F_nF_{n-(k-1)}-F_k & \text{if } n \text{ is even.}\end{cases}$$

For the case when we multiply the i^{th} row of $\mathcal{F}_n$ and the i^{th} column of $\mathcal{F}_n$, we have the famous formula (2). Also, formula (2) is the case when $k=0$ in Corollary 2.10.

3. EIGENVALUES OF $\mathcal{Q}_n$

In this section, we consider the eigenvalues of $\mathcal{Q}_n$.

Let $\mathcal{D}=\{\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n : x_1\ge x_2\ge\cdots\ge x_n\}$. For $\mathbf{x},\mathbf{y}\in\mathcal{D}$, $\mathbf{x}\prec\mathbf{y}$ if $\sum_{i=1}^k x_i\le\sum_{i=1}^k y_i$, $k=1,2,\ldots,n$ and if $k=n$, then the equality holds. When $\mathbf{x}\prec\mathbf{y}$, $\mathbf{x}$ is said to be *majorized* by $\mathbf{y}$, or $\mathbf{y}$ is said to *majorize* $\mathbf{x}$. The condition for majorization can be rewritten as follows: for $\mathbf{x},\mathbf{y}\in\mathcal{D}$, $\mathbf{x}\prec\mathbf{y}$ if $\sum_{i=0}^k x_{n-i}\ge\sum_{i=0}^k y_{n-i}$, $k=0,1,\ldots,n-2$, and if $k=n-1$, then equality holds.

The following is an interesting simple fact:

$$(\bar{x},\ldots,\bar{x})\prec(x_1,\ldots,x_n),\text{ where } \bar{x}=\frac{\sum_{n=1}^n x_i}{n}.$$

More interesting facts about majorizations can be found in [4].

An $n\times n$ matrix $P=[p_{ij}]$ is *doubly stochastic* if $p_{ij}\ge 0$ for $i,j=1,2,\ldots,n$, $\sum_{i=1}^n p_{ij}=1$, $j=1,2,\ldots,n$, and $\sum_{j=1}^n p_{ij}=1$, $i=1,2,\ldots,n$. In 1929, Hardy, Littlewood, and Polya proved that a necessary and sufficient condition that $\mathbf{x}\prec\mathbf{y}$ is that there exist a doubly stochastic matrix P such that $\mathbf{x}=\mathbf{y}P$.

We know both the eigenvalues and the main diagonal elements of a real symmetrix matrix are real numbers. The precise relationship between the main diagonal elements and the eigenvalues is given by the notion of majorization as follows: the vector of eigenvalues of a symmetrix matrix is majorized by the diagonal elements of the matrix.

Note that $\det\mathcal{F}_n=1$ and $\det\mathcal{Q}_n=1$. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of $\mathcal{Q}_n$. Since $\mathcal{Q}_n=\mathcal{F}_n\mathcal{F}_n^T$ and $\sum_{i=1}^k F_i^2=F_{k+1}F_k$, the eigenvalues of $\mathcal{Q}_n$ are all positive and

$$(F_{n+1}F_n, F_nF_{n-1},\ldots,F_2F_1)\prec(\lambda_1,\lambda_2,\ldots,\lambda_n).$$

In [1], we find the interesting combinatorial property, $\sum_{i=0}^n\binom{n-i}{i}=F_{n+1}$. So we have the following corollaries.

Corollary 3.1: Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be the eigenvalues of $\mathcal{Q}_n$. Then

$$\lambda_1+\lambda_2+\cdots+\lambda_n=\begin{cases}\left(\sum_{i=0}^n\binom{n-i}{i}\right)^2-1 & \text{if } n \text{ is odd,}\\ \left(\sum_{i=0}^n\binom{n-i}{i}\right)^2 & \text{if } n \text{ is even.}\end{cases}$$

Proof: Since $(F_{n+1}F_n, F_nF_{n-1}, \dots, F_2F_1) \prec (\lambda_1, \lambda_2, \dots, \lambda_n)$, and from Corollary 2.10,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \begin{cases} (F_{n+1})^2 - F_1 & \text{if } n \text{ is odd,} \\ (F_{n+1})^2 & \text{if n is even,} \end{cases} = \begin{cases} \left(\sum_{i=0}^{n}\binom{n-i}{i}\right)^2 - 1 & \text{if } n \text{ is odd,} \\ \left(\sum_{i=0}^{n}\binom{n-i}{i}\right)^2 & \text{if } n \text{ is even.} \end{cases} \quad \square$$

Corollary 3.2: If n is an odd number, then

$$n\lambda_n \le \left(\sum_{i=0}^{n}\binom{n-i}{i}\right)^2 - 1 \le n\lambda_1.$$

If n is an even number, then

$$n\lambda_n \le \left(\sum_{i=0}^{n}\binom{n-i}{i}\right)^2 \le n\lambda_1.$$

Proof: Let $s_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. Since

$$\left(\frac{s_n}{n}, \frac{s_n}{n}, \dots, \frac{s_n}{n}\right) \prec (\lambda_1, \lambda_2, \dots, \lambda_n),$$

we have $\lambda_n \le \frac{s_n}{n} \le \lambda_1$. Therefore, the proof is complete. $\square$

From equation (5), we have

$$(3, 3, \dots, 3, 2, 1) \prec \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \dots, \frac{1}{\lambda_1}\right). \tag{6}$$

Thus, there exists a doubly stochastic matrix $T = [t_{ij}]$ such that

$$(3, 3, \dots, 3, 2, 1) = \left(\frac{1}{\lambda_n}, \frac{1}{\lambda_{n-1}}, \dots, \frac{1}{\lambda_1}\right)\begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{21} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{bmatrix}.$$

That is, we have $\frac{1}{\lambda_n}t_{1n} + \frac{1}{\lambda_{n-1}}t_{2n} + \cdots + \frac{1}{\lambda_1}t_{nn} = 1$ and $t_{1n} + t_{2n} + \cdots + t_{nn} = 1$.

Lemma 3.3: For each $i = 1, 2, \dots, n$, $t_{n-(i-1), n} \le \frac{\lambda_i}{n-1}$.

Proof: Suppose that $t_{n-(i-1), n} > \frac{\lambda_i}{n-1}$. Then

$$t_{1n} + t_{2n} + \cdots + t_{nn} > \frac{\lambda_1}{n-1} + \frac{\lambda_2}{n-1} + \cdots \frac{\lambda_n}{n-1} = \frac{1}{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n).$$

Since $t_{1n} + t_{2n} + \cdots + t_{nn} = 1$ and $\sum_{i=1}^{n}\lambda_i \ge n$, this yields a contradiction, so $t_{n-(i-1), n} \le \frac{\lambda_i}{n-1}$. $\square$

From Lemma 3.3, we have $1 - (n-1)\frac{1}{\lambda_i}t_{n-(i-1), n} \ge 0$. Let $\alpha = s_n - (n-1)$. Therefore, we have the following theorem.

Theorem 3.4: For $(\alpha, 1, 1, \dots, 1) \in \mathcal{D}$, $(\alpha, 1, 1, \dots, 1) \prec (\lambda_1, \lambda_2, \dots, \lambda_n)$.

Proof: A necessary and sufficient condition that $(\alpha, 1, 1, \dots, 1) \prec (\lambda_1, \lambda_2, \dots, \lambda_n)$ is that there exist a doubly stochastic matrix P such that $(\alpha, 1, 1, \dots, 1) = (\lambda_1, \lambda_2, \dots, \lambda_n)P$.

We define an $n \times n$ matrix $P = [p_{ij}]$ as follows:

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{12} \\ p_{21} & p_{22} & \cdots & p_{22} \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{n2} \end{bmatrix},$$

where $p_{i2} = \frac{1}{\lambda_i} t_{n-(i-1),n}$ and $p_{i1} = 1-(n-1)p_{i2}$, $i = 1, 2, \ldots, n$. Since T is doubly stochastic and $\lambda_i > 0$, $p_{i2} \geq 0$, $i = 1, 2, \ldots, n$. By Lemma 3.3, $p_{i1} \geq 0$, $i = 1, 2, \ldots, n$. Then

$$p_{12} + p_{22} + \cdots + p_{n2} = \frac{t_{nn}}{\lambda_1} + \frac{t_{n-1,n}}{\lambda_2} + \cdots + \frac{t_{1n}}{\lambda_n} = 1,$$

$$p_{i1} + (n-1)p_{i2} = 1-(n-1)p_{i2} + (n-1)p_{i2} = 1,$$

and

$$\begin{aligned} p_{11} + p_{21} + \cdots + p_{n1} &= 1-(n-1)p_{12} + 1-(n-1)p_{22} + \cdots + 1-(n-1)p_{n2} \\ &= n - n(p_{12} + p_{22} + \cdots + p_{n2}) + p_{12} + p_{22} + \cdots + p_{n2} = 1. \end{aligned}$$

Thus, p is a doubly stochastic matrix. Furthermore,

$$\begin{aligned} \lambda_1 p_{12} + \lambda_2 p_{22} + \cdots + \lambda_n p_{n2} &= \lambda_1 \frac{t_{nn}}{\lambda_1} + \lambda_2 \frac{t_{n-1,n}}{\lambda_2} + \cdots + \lambda_n \frac{t_{1n}}{\lambda_n} \\ &= t_{nn} + t_{n-1,n} + \cdots + t_{1n} = 1 \end{aligned}$$

and

$$\begin{aligned} \lambda_1 p_{11} + \lambda_2 p_{21} + \cdots + \lambda_n p_{n1} &= \lambda_1(1-(n-1)p_{12}) + \cdots + \lambda_n(1-(n-1)p_{n2}) \\ &= \lambda_1 + \lambda_2 + \cdots + \lambda_n - (n-1)(\lambda_1 p_{12} + \lambda_2 p_{22} + \cdots + \lambda_n p_{n2}) \\ &= \lambda_1 + \lambda_2 + \cdots + \lambda_n - (n-1) = \alpha. \end{aligned}$$

Thus, $(\alpha, 1, 1, \ldots, 1) = (\lambda_1, \lambda_2, \ldots, \lambda_n)P$, so $(\alpha, 1, 1, \ldots, 1) \prec (\lambda_1, \lambda_2, \ldots, \lambda_n)$. □

From equation (6), we have the following lemma.

Lemma 3.5: For $k = 2, 3, \ldots, n$, $\lambda_k \geq \frac{1}{3(k-1)}$.

Proof: From (6), for $k \geq 2$,

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_k} \leq 1 + 2 + 3 + \cdots + 3 = 3(k-1).$$

Thus,

$$\frac{1}{\lambda_k} \leq 3(k-1) - \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_{k-1}}\right) \leq 3(k-1).$$

Therefore, for $k = 2, 3, \ldots, n$, $\lambda_k \geq \frac{1}{3(k-1)}$. □

Corollary 3.6: For $k = 1, 2, \ldots, n-2$, $\lambda_{n-k} \leq (k+1) - \frac{n-k}{3(n-1)}$. In particular, $\alpha \leq \lambda_1$ and $\frac{1}{3(k-1)} \leq \lambda_n \leq \frac{1}{3}$.

Proof: If $k = 1$, then $\lambda_n + \lambda_{n-1} \leq 2$. By Lemma 3.5, we have $\lambda_{n-1} \leq 2 - \frac{1}{3(n-1)}$. Hence, by induction on n, the proof is complete for $k = 1, 2, \ldots, n-2$. In particular, by Theorem 3.4 and (6), $\frac{1}{3(n-1)} \leq \lambda_n \leq \frac{1}{3}$. □

Since $\det \mathfrak{Q}_n = \lambda_1\lambda_2\ldots\lambda_n = 1$, $\lambda_2\lambda_3\ldots\lambda_n = \frac{1}{\lambda_1}$, we have $\lambda_1^{n-1} \geq \lambda_1\ldots\lambda_{n-1} = \frac{1}{\lambda_n}$. Thus,

$$\lambda_n \geq \left(\frac{1}{\lambda_1}\right)^{n-1}.$$

Therefore,

$$\left(\frac{1}{\lambda_1}\right)^{n-1} \leq \lambda_n \leq \frac{1}{3}.$$

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