PARTITION FORMS OF FIBONACCI NUMBERS

Sun Ping
Dept. of Mathematics, Northeastern University, Shenyang, 110006, P.R. China
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In the notation of Comtet [1], define the partitions of integer n as $n = \sum ik_i$, where $i \geq 1$ is a summand and $k_i \geq 0$ is the frequency of summand i. It is known that the number of subsets of an n-element set is 2^n and

$$2^n = \sum_{\sum ik_i = n+1} \frac{(\sum k_i)!}{\prod k_i!}, \tag{1}$$

because of

$$\sum_{\substack{\sum ik_i = n+1 \\ \sum k_i = k}} \frac{1}{\prod k_i!} = \frac{1}{k!}\binom{n}{k-1}.$$

Equation (1) shows that the number of subsets of an n-element set is related to the number of summands in partitions of n. It is surprising that the sums on the right of identity (1) become Fibonacci numbers when some summands of the partitions of n no longer appear.

By means of generating functions, this article obtains the following result.

Theorem: For any $n \geq 1$, Fibonacci numbers satisfy

(a)
$$F_n = \sum_{\substack{\sum ik_i = n+1 \\ k_1 = 0}} \frac{(\sum k_i)!}{\prod k_i!}, \tag{2}$$

(b)
$$F_n = \sum_{\substack{\sum ik_i = n \\ \text{all } k_{2i} = 0}} \frac{(\sum k_i)!}{\prod k_i!}. \tag{3}$$

For example, the partitions of the integer 7 are

7, 1+6, 2+5, 3+4, 1+1+5, 1+2+4, 1+3+3, 2+2+3, 1+1+1+4, 1+1+2+3,
1+2+2+2, 1+1+1+1+3, 1+1+1+2+2, 1+1+1+1+1+2, 1+1+1+1+1+1+1.

From (2), we have

$$F_6 = \frac{1!}{1!} + \frac{2!}{1!\cdot 1!} + \frac{2!}{1!\cdot 1!} + \frac{3!}{2!\cdot 1!} = 1+2+2+3 = 8,$$

and from (3),

$$F_7 = \frac{1!}{1!} + \frac{3!}{2!\cdot 1!} + \frac{3!}{1!\cdot 2!} + \frac{5!}{4!\cdot 1!} + \frac{7!}{7!} = 1+3+3+5+7 = 13.$$

The Theorem can be proved easily by using the recurrence relations of Fibonacci numbers and the results of Bell polynomials $B_{n,k}$ [1]:

$$\frac{1}{k!}\left(\sum_{m\geq 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n\geq k} B_{n,k} \frac{t^n}{n!}, \quad k = 0, 1, 2, \ldots,$$

and

$$\frac{1}{n!}B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) = \sum_{\substack{\Sigma ik_i=n \\ \Sigma k_i=k,\, k_i\ge 0}} \frac{\prod x_i^{k_i}}{\prod k_i!(i!)^{k_i}}.$$

In this article, $[t^n]f(t)$ means the coefficient of t^n is in the formal series $f(t)$, so that

$$\sum_{n\ge 1} F_n t^n = \frac{t}{1-t-t^2} \quad \text{can be written as} \quad F_n = [t^n]\frac{t}{1-t-t^2}.$$

Proof of Theorem: (a) It is well known that $F_{n+2} = F_n + F_{n+1}$, $n \ge 1$, then

$$\begin{aligned} F_n &= [t^{n+2}]\frac{t}{1-t-t^2} - [t^{n+1}]\frac{t}{1-t-t^2} \\ &= [t^{n+1}]\frac{1-t}{1-t-t^2} = [t^{n+1}]\frac{1}{1-\left(\frac{t^2}{1-t}\right)} = [t^{n+1}]\frac{1}{1-(t^2+t^3+t^4+\cdots)} \\ &= \sum_{k\ge 1}[t^{n+1}](t^2+t^3+t^4+\cdots)^k = \sum_{k\ge 1}\sum_{\substack{\Sigma ik_i=n+1 \\ k_1=0,\, \Sigma k_i=k}}\left[\frac{k!}{\prod_{i\ge 1}k_i!}\right] = \sum_{\substack{\Sigma ik_i=n+1 \\ k_1=0}}\frac{(\Sigma k_i)!}{\prod k_i!}. \end{aligned}$$

(b) The proof is similar; notice that $F_n = F_{n+1} - F_{n-1}$, $n \ge 2$. Thus, for any $n \ge 2$,

$$\begin{aligned} F_n &= [t^{n+1}]\frac{t}{1-t-t^2} - [t^{n-1}]\frac{t}{1-t-t^2} \\ &= [t^n]\frac{1-t^2}{1-t-t^2} = [t^n]\frac{1}{1-\left(\frac{t}{1-t^2}\right)} = [t^n]\frac{1}{1-(t+t^3+t^5+t^7\cdots)} \\ &= \sum_{k\ge 1}[t^n](t+t^3+t^5+t^7\cdots)^k = \sum_{k\ge 1}\sum_{\substack{\Sigma ik_i=n \\ \text{all } k_{2i}=0,\, \Sigma k_i=k}}\left[\frac{k!}{\prod_{i\ge 1}k_i!}\right] = \sum_{\substack{\Sigma ik_i=n \\ \text{all } k_{2i}=0}}\frac{(\Sigma k_i)!}{\prod k_i!}. \end{aligned}$$

Remark 1: The number of summands on the right of (2) is $p(n+1) - p(n)$, and that of (3) is $q(n)$. Here, $p(n)$ is the number of partitions of n and $q(n)$ is the number of partitions of n into distinct summands, see [1].

Remark 2: It is well known that Fibonacci numbers have a simple combinatorial meaning, F_n is the number of subsets of $\{1, 2, 3, \ldots, n\}$ such that no two elements are adjacent. Comparing with (1), the Theorem shows that Fibonacci numbers have a kind of new combinatorial structure as a weighted sum over partitions.

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REFERENCES

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