SOME CONSEQUENCES OF GAUSS' TRIANGULAR NUMBER THEOREM

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INTRODUCTION

If n is a nonnegative integer, let $t(n) = n(n+1)/2$ denote the n^{th} triangular number. Gauss' triangular number theorem states that if x is a complex variable such that $|x| < 1$, then

$$\prod_{n\geq 1} \frac{1-x^{2n}}{1-x^{2n-1}} = \sum_{n\geq 0} x^{t(n)}$$

(see [1], p. 326, Ex. 5b, or [3], Theorem 354, p. 284).

In this note, we make use of this Gaussian formula to derive several apparently new identities concerning $q_0(n)$, the number of self-conjugate partitions of n.

PRELIMINARIES

Definition 1: Let $p(n)$ denote the number of unrestricted partitions of n.

Definition 2: Let $q_0(n)$ denote the number of partitions of n into distinct odd parts (or the number of self-conjugate partitions of n).

Definition 3: If $r \geq 1$, let $q_r(n)$ denote the number of partitions of n into distinct parts in r colors.

Remark: If $f(n)$ is any of the above partition functions, we define $f(0) = 1$, $f(\alpha) = 0$ if α is not a nonnegative integer.

Definition 4 (Pentagonal numbers): If $k \in \mathbb{Z}$, then

$$\omega(k) = \frac{k(3k-1)}{2}.$$

IDENTITIES

Let x be a complex variable such that $|x| < 1$. Let $r \geq 1$. Let $j \geq 1$. Then we have:

$$\prod_{n\geq 1} \frac{1-x^{2n}}{1-x^{2n-1}} = \sum_{n\geq 0} x^{t(n)}, \tag{1}$$

$$\prod_{n\geq 1} (1-x^{jn})^{-1} = \sum_{n\geq 0} p\left(\frac{n}{j}\right) x^n, \tag{2}$$

$$\prod_{n\geq 1} (1-x^{jn}) = 1 + \sum_{k\geq 1} (-1)^k (x^{j\omega(k)} + x^{j\omega(-k)}), \tag{3}$$

$$\prod_{n\geq 1} (1+x^{2n-1}) = \sum_{n\geq 0} q_0(n) x^n, \tag{4}$$

$$\prod_{n\geq 1}(1-x^{2n-1})^{-1}=\prod_{n\geq 1}(1+x^n)=\sum_{n\geq 0}q(n)x^n, \tag{5}$$

$$\prod_{n\geq 1}(1+x^n)^r=\sum_{n\geq 0}q_r(n)x^n, \tag{6}$$

$$\left(\sum_{n\geq 0}a_nx^n\right)\left(\sum_{n\geq 0}b_nx^n\right)=\sum_{n\geq 0}\left(\sum_{k=0}^{n}a_{n-k}b_k\right)x^n. \tag{7}$$

Remark: For proofs, see Chapter 19 of [3].

THE MAIN RESULTS

Theorem 1: Let the integer $n\geq 0$. Then

$$q_0(n)+\sum_{k\geq 1}(-1)^k(q_0(n-4\omega(k))+q_0(n-4\omega(-k)))=\begin{cases}1 & \text{if } n=t(j) \text{ for some } j\geq 0,\\ 0 & \text{otherwise.}\end{cases}$$

Proof: By (1) and (5), we have:

$$\sum_{n\geq 0}x^{t(n)}=\prod_{n\geq 1}(1+x^n)(1-x^{2n})=\prod_{n\geq 1}(1+x^{2n-1})(1+x^{2n})(1-x^{2n})$$

$$=\prod_{n\geq 1}(1+x^{2n-1})(1-x^{4n})=\prod_{n\geq 1}(1+x^{2n-1})\prod_{n\geq 1}(1-x^{4n})=\left(\sum_{n\geq 0}q_0(n)x^n\right)\prod_{n\geq 1}(1-x^{4n})$$

$$=\sum_{n\geq 0}\left(q_0(n)+\sum_{k\geq 1}(q_0(n-4\omega(k))+q_0(n-4\omega(-k))\right)x^n.$$

The last few steps required the use of (4), (3), and (7). The conclusion now follows by matching coefficients of like powers of x.

Remark: We earlier proved similar recurrences concerning $q_0(n)$, namely:

$$q_0(n)+\sum_{k\geq 1}(-1)^k(q_0(n-\omega(k))+q_0(n-\omega(-k)))=\begin{cases}2(-1)^m & \text{if } n=2m^2,\\ 0 & \text{otherwise.}\end{cases}$$

$$q_0(n)+\sum_{k\geq 1}(-1)^k(q_0(n-2\omega(k))+q_0(n-2\omega(-k)))=\begin{cases}(-1)^{\left[\frac{1\pm m}{2}\right]} & \text{if } n=\omega(\pm m),\\ 0 & \text{otherwise.}\end{cases}$$

(See Theorem 2 in each of [4] and [5], respectively.)

Theorem 2: Let the integer $n\geq 0$. Then

$$q_0(n)+\sum_{k\geq 1}(q_0(n-8\omega(k))+q_0(n-8\omega(-k)))=\sum_{j\geq 0}q\left(\frac{n-t(j)}{4}\right).$$

Proof: In the proof of Theorem 1, we encountered the identity:

$$\prod_{n\geq 1}(1+x^{2n-1})(1-x^{4n})=\sum_{n\geq 0}x^{t(n)}.$$

Therefore,

$$\prod_{n\geq 1}(1+x^{2n-1})(1-x^{4n})(1+x^{4n})=\prod_{n\geq 1}(1+x^{4n})\sum_{n\geq 0}x^{t(n)},$$

$$\prod_{n\geq 1}(1+x^{2n-1})\prod_{n\geq 1}(1-x^{8n})=\left(\sum_{n\geq 0}q\left(\frac{n}{4}\right)x^n\right)\left(\sum_{n\geq 0}x^{t(n)}\right),$$

$$\left(\sum_{n\geq 0}q_0(n)x^n\right)\prod_{n\geq 1}(1-x^{8n})=\left(\sum_{n\geq 0}q\left(\frac{n}{4}\right)x^n\right)\left(\sum_{n\geq 0}x^{t(n)}\right).$$

The conclusion now follows by invoking (4), (3), and (7), and matching coefficients of like powers of x.

The following theorem regarding $q_0(n)$ is not a recurrence; it expresses $q_0(n)$ in terms of $p(n)$.

Theorem 3:

$$q_0(n)=\sum_{j\geq 0}p\left(\frac{n-t(j)}{4}\right).$$

Proof:

$$\sum_{n\geq 0}q_0(n)x^n=\prod_{n\geq 1}(1+x^{2n-1})=\prod_{n\geq 1}\frac{1+x^n}{1+x^{2n}}=\prod_{n\geq 1}\frac{1+x^{2n}}{(1-x^{4n})(1-x^{2n-1})}$$

$$=\prod_{n\geq 1}(1-x^{4n})^{-1}\prod_{n\geq 1}\frac{1-x^{2n}}{1-x^{2n-1}}=\left(\sum_{n\geq 0}p\left(\frac{n}{4}\right)x^n\right)\left(\sum_{n\geq 0}x^{t(n)}\right)$$

by (4), (3), and (1). The conclusion now follows if one invokes (7) and matches coefficients of like powers of x.

Remark: Theorem 3 is essentially Watson's identity:

$$\chi(x)=\left(\sum_{n=0}^{\infty}x^{\frac{n(n+1)}{2}}\right)\left(\sum_{n=0}^{\infty}p(n)x^{4n}\right)$$

(see [6], p. 551).

The content of Theorem 3 may be stated more explicitly as Theorem 3a below.

Theorem 3a:

$$q_0(4n)=p(n)+p(n-7)+p(n-9)+p(n-30)+p(n-34)+\cdots,$$
$$q_0(4n+1)=p(n)+p(n-5)+p(n-11)+p(n-26)+p(n-38)+\cdots,$$
$$q_0(4n+2)=p(n-1)+p(n-2)+p(n-16)+p(n-19)+p(n-47)+\cdots,$$
$$q_0(4n+3)=p(n)+p(n-3)+p(n-13)+p(n-22)+p(n-42)+\cdots.$$

Corollary: $q_0(n)\geq p([n/4])$.

Proof: This follows from Theorem 3.

Remark: In [2], J. Ewell proved a theorem similar to Theorem 3, namely:

$$q(n)=\sum_{j\geq 0}p\left(\frac{n-t(j)}{2}\right).$$

Using similar reasoning, it follows that

$$q_2(n) = \sum_{j \geq 0} p(n - t(j)).$$

REFERENCES

1. T. Apostol. *Introduction to Analytic Number Theory*. New York: Springer-Verlag, 1976.
2. J. Ewell. "Partition Recurrences." *J. Combinatorial Theory Ser. A* **14** (1973):125-27.
3. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*. 4th ed. Oxford: Oxford University Press, 1960.
4. K. Ono, N. Robbins, & B. Wilson. "Some Recurrences for Arithmetic Functions." *J. Indian Math. Soc.* **62** (1996):29-51.
5. N. Robbins. "Some New Partition Identities." *Ars Combinatoria* **50** (1998):193-206.
6. G. N. Watson. "Two Tables of Partitions." *Proc. London Math. Soc.* **42.2** (1937):550-56.

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