ON THE ALMOST HILBERT-SMITH MATRICES

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1. INTRODUCTION

The study of GCD matrices was initiated by Beslin and Ligh [5]. In that paper the authors investigated GCD matrices in the direction of their structure, determinant, and arithmetic in Z_n. The determinants of GCD matrices were investigated in [6] and [11]. Furthermore, many other results on GCD matrices were established or conjectured (see [2]-[4], [7]-[10], and [12]).

In this paper we define an $n\times n$ matrix $S=(s_{ij})$, where $s_{ij}=\frac{(i,j)}{ij}$, and call S the "almost Hilbert-Smith matrix." In the second section we calculate the determinant and the inverse of the almost Hilbert-Smith matrix. In the last section we consider a generalization of the almost Hilbert-Smith matrix.

2. THE STRUCTURE OF THE ALMOST HILBERT-SMITH MATRIX

The $n\times n$ matrix $S=(s_{ij})$, where $s_{ij}=\frac{(i,j)}{ij}$, is called the almost Hilbert-Smith matrix. In this section we present a structure theorem and then calculate the value of the determinant of the almost Hilbert-Smith matrix. The following theorem describes the structure of the almost Hilbert-Smith matrix.

Theorem 1: Let $S=(s_{ij})$ be the $n\times n$ almost Hilbert-Smith matrix. Define the $n\times n$ matrix $A=(a_{ij})$ by

$$a_{ij}=\begin{cases}\frac{\sqrt{\phi(j)}}{i} & \text{if } j\mid i,\\ 0 & \text{otherwise,}\end{cases}$$

where ϕ is Euler's totient function. Then $S=AA^T$.

Proof: The ij-entry in AA^T is

$$(AA^T)_{ij}=\sum_{k=1}^{n}a_{ik}a_{jk}=\sum_{\substack{k\mid i\\ k\mid j}}\frac{\sqrt{\phi(k)}}{i}\frac{\sqrt{\phi(k)}}{j}=\frac{1}{ij}\sum_{k\mid(i,j)}\phi(k)=\frac{(i,j)}{ij}=s_{ij}.\ \square$$

Corollary 1: The almost Hilbert-Smith matrix is positive definite, and hence invertible.

Proof: The matrix $A=(a_{ij})$ is a lower triangular matrix and its diagonal is

$$\left(\frac{\sqrt{\phi(1)}}{1},\frac{\sqrt{\phi(2)}}{2},\ldots,\frac{\sqrt{\phi(n)}}{n}\right).$$

It is clear that $\det A=\frac{1}{n!}[\phi(1)\phi(2)\ldots\phi(n)]^{1/2}$ and $\phi(i)>0$ for $1\le i\le n$. Since $\det A>0$, $rank(S)=rank(AA^T)=rank(A)=n$. Thus, S is positive definite. $\square$

Corollary 2: If S is the $n \times n$ almost Hilbert-Smith matrix, then

$$\det S = \frac{1}{(n!)^2}\phi(1)\phi(2)\ldots\phi(n).$$

Proof: By Theorem 1, and since the matrix A is a lower triangular matrix, the result is immediate. □

The matrix A in Theorem 1 can be written as $A = E\Lambda^{1/2}$, where the $n \times n$ matrices $E = (e_{ij})$ and $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ are given by

$$e_{ij} = \begin{cases} \frac{1}{i} & \text{if } j \mid i, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

and $\lambda_j = \phi(j)$. Thus, $S = AA^T = (E\Lambda^{1/2})(E\Lambda^{1/2})^T = E\Lambda E^T$.

Theorem 2: Let $S = (s_{ij})$ be the $n \times n$ almost Hilbert-Smith matrix. Then the inverse of S is the matrix $B = (b_{ij})$ such that

$$b_{ij} = ij\sum_{\substack{i \mid k \\ j \mid k}} \frac{1}{\phi(k)}\mu\left(\frac{k}{i}\right)\mu\left(\frac{k}{j}\right),$$

where μ denotes the Möbius function.

Proof: Let $E = (e_{ij})$ be the matrix defined in (1) and the $n \times n$ matrix $U = (u_{ij})$ be defined as follows:

$$u_{ij} = \begin{cases} j\mu\left(\frac{i}{j}\right) & \text{if } j \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Calculating the ij-entry of the product EU gives

$$(EU)_{ij} = \sum_{k=1}^{n} e_{ik}u_{kj} = \sum_{\substack{k \mid i \\ j \mid k}} \frac{1}{i} j\mu\left(\frac{k}{j}\right) = \frac{j}{i}\sum_{k \mid \frac{i}{j}} \mu(k) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence, $U = E^{-1}$. If $\Lambda = diag(\phi(1), \phi(2), \ldots, \phi(n))$, then $S = E\Lambda E^T$. Thus, $S^{-1} = U^T\Lambda^{-1}U = (b_{ij})$, where

$$b_{ij} = (U^T\Lambda^{-1}U)_{ij} = \sum_{k=1}^{n} \frac{1}{\phi(k)}u_{ki}u_{kj} = ij\sum_{\substack{i \mid k \\ j \mid k}} \frac{1}{\phi(k)}\mu\left(\frac{k}{i}\right)\mu\left(\frac{k}{j}\right). \quad \square$$

Example 1: Let $S = (s_{ij})$ be the 4×4 almost Hilbert-Smith matrix,

$$S = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{6} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{12} & \frac{1}{4} \end{bmatrix}.$$

By Theorem 2, $S^{-1} = (b_{ij})$, where

$$b_{11} = 1\cdot 1\cdot\left(\frac{\mu(1)\mu(1)}{\phi(1)} + \frac{\mu(2)\mu(2)}{\phi(2)} + \frac{\mu(3)\mu(3)}{\phi(3)} + \frac{\mu(4)\mu(4)}{\phi(4)}\right) = \frac{5}{2},$$

$$b_{12} = 1\cdot 2\cdot\left(\frac{\mu(2)\mu(1)}{\phi(2)} + \frac{\mu(4)\mu(2)}{\phi(4)}\right) = -2,\quad b_{13} = 1\cdot 3\cdot\frac{\mu(3)\mu(1)}{\phi(3)} = -\frac{3}{2},$$

$$b_{14} = 1\cdot 4\cdot\frac{\mu(4)\mu(1)}{\phi(4)} = 0,\quad b_{22} = 2\cdot 2\cdot\left(\frac{\mu(1)\mu(1)}{\phi(2)} + \frac{\mu(2)\mu(2)}{\phi(4)}\right) = 6,\quad b_{23} = 0,$$

$$b_{24} = 2\cdot 4\cdot\frac{\mu(2)\mu(1)}{\phi(4)} = -4,\quad b_{33} = 3\cdot 3\cdot\frac{\mu(1)\mu(1)}{\phi(3)} = \frac{9}{2},\quad b_{34} = 0,\quad b_{44} = 4\cdot 4\cdot\frac{\mu(1)\mu(1)}{\phi(4)} = 8.$$

Therefore, since S^{-1} is symmetric, we have

$$S^{-1} = \begin{bmatrix} \frac{5}{2} & -2 & -\frac{3}{2} & 0 \\ -2 & 6 & 0 & -4 \\ -\frac{3}{2} & 0 & \frac{9}{2} & 0 \\ 0 & -4 & 0 & 8 \end{bmatrix}.$$

3. GENERALIZATION OF THE ALMOST HILBERT-SMITH MATRIX

In this section we consider an $n\times n$ matrix, the ij-entry of which is the positive m^{th} power of the ij-entry of the almost Hilbert-Smith matrix:

$$s_{ij}^m = \frac{(i,j)^m}{i^m j^m}.$$

Let m be a positive integer and let $S = (s_{ij})$ be the $n\times n$ almost Hilbert-Smith matrix. Define an $n\times n$ matrix S^m, the ij-entry of which is s_{ij}^m. Then

$$s_{ij}^m = \frac{(i,j)^m}{i^m j^m} = \sum_{k\mid(i,j)} \frac{J_m(k)}{i^m j^m},$$

where J_m is Jordan's generalization of Euler's totient function [1], given by

$$J_m(k) = \sum_{e\mid k} e^m \mu\left(\frac{k}{e}\right).$$

Theorem 3: Let $C = (c_{ij})$ be an $n\times n$ matrix defined by

$$c_{ij} = \begin{cases} \dfrac{\sqrt{J_m(j)}}{i^m} & \text{if } j\mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Then $S^m = CC^T$.

Proof: The ij-entry in CC^T is

$$(CC^T)_{ij} = \sum_{k=1}^{n} c_{ik}c_{jk} = \sum_{\substack{k\mid i\\ k\mid j}} \frac{\sqrt{J_m(k)}}{i^m}\frac{\sqrt{J_m(k)}}{j^m}$$

$$= \frac{1}{i^m j^m}\sum_{k\mid(i,j)} J_m(k) = \frac{(i,j)^m}{i^m j^m} = s_{ij}^m. \quad \square$$

Corollary 3: The matrix $S^m = (s_{ij}^m)$ is positive definite, and hence invertible.

Proof: The matrix $C = (c_{ij})$ is a lower triangular matrix and its diagonal is

$$\left(\frac{\sqrt{J_m(1)}}{1^m}, \frac{\sqrt{J_m(2)}}{2^m}, \ldots, \frac{\sqrt{J_m(n)}}{n^m}\right).$$

It is clear that

$$\det C = \frac{1}{(n!)^m}[J_m(1)J_m(2)\ldots J_m(n)]^{1/2}$$

and $J_m(i) > 0$ for $1 \le i \le n$. Since $\det C > 0$, $rank(S^m) = rank(CC^T) = rank(C) = n$. Thus, S^m is positive definite. □

Corollary 4: If $S^m = (s_{ij}^m)$ is the $n \times n$ matrix whose ij-entry is $s_{ij}^m = \frac{(i,j)^m}{i^m j^m}$, then

$$\det S^m = \frac{1}{(n!)^{2m}} J_m(1) J_m(2) \ldots J_m(n).$$

Proof: By Theorem 3, and since the matrix C is a lower triangular matrix, the result is immediate. □

Example 2: Consider S^3, where S is the 5×5 almost Hilbert-Smith matrix. Then

$$S^3 = \begin{bmatrix} 1 & \frac{1}{8} & \frac{1}{27} & \frac{1}{64} & \frac{1}{125} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{216} & \frac{1}{64} & \frac{1}{1000} \\ \frac{1}{27} & \frac{1}{216} & \frac{1}{27} & \frac{1}{1728} & \frac{1}{3375} \\ \frac{1}{64} & \frac{1}{64} & \frac{1}{1728} & \frac{1}{64} & \frac{1}{8000} \\ \frac{1}{125} & \frac{1}{1000} & \frac{1}{3375} & \frac{1}{8000} & \frac{1}{125} \end{bmatrix}.$$

By Corollary 4, we have

$$\det S^3 = \frac{1}{(5!)^6} J_3(1) J_3(2) J_3(3) J_3(4) J_3(5) = \frac{19747}{46656000000}. \quad \square$$

We now define the $n \times n$ matrices $D = (d_{ij})$ and $\Omega = diag(\omega_1, \omega_2, \ldots, \omega_n)$ by

$$d_{ij} = \begin{cases} \frac{1}{i^m} & \text{if } j \mid i, \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and $\omega_j = J_m(j)$. Then the matrix $C = (c_{ij})$ can be written as $C = D\Omega^{1/2}$. Thus, we have

$$S^m = CC^T = (D\Omega^{1/2})(D\Omega^{1/2})^T = D\Omega D^T.$$

Theorem 4: The inverse of the matrix $S^m = (s_{ij}^m)$ is the matrix $G = (g_{ij})$, where

$$g_{ij} = i^m j^m \sum_{\substack{i \mid k \\ j \mid k}} \frac{1}{J_m(k)} \mu\left(\frac{k}{i}\right) \mu\left(\frac{k}{j}\right).$$

Proof: Let $D = (d_{ij})$ be the matrix defined in (2) and the $n \times n$ matrix $V = (v_{ij})$ be defined as follows:

$$v_{ij} = \begin{cases} j^m \mu\left(\frac{i}{j}\right) & \text{if } j \mid i, \\ 0 & \text{otherwise.} \end{cases}$$

Calculating the ij-entry of the product DV gives

$$(DV)_{ij} = \sum_{k=1}^{n} d_{ik} v_{kj} = \sum_{\substack{k \mid i \\ j \mid k}} \frac{1}{i^m} j^m \mu\left(\frac{k}{j}\right)$$

$$= \frac{j^m}{i^m} \sum_{k \mid \frac{i}{j}} \mu(k) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence, $V = D^{-1}$. If $\Omega = diag(J_m(1), J_m(2), \ldots, J_m(n))$, then $S^m = D\Omega D^T$. Therefore, $(S^m)^{-1} = V^T \Omega^{-1} V = G = (g_{ij})$, where

$$g_{ij} = (V^T \Omega^{-1} V)_{ij} = \sum_{k=1}^{n} \frac{1}{J_m(k)} v_{ki} v_{kj} = i^m j^m \sum_{\substack{i \mid k \\ j \mid k}} \frac{1}{J_m(k)} \mu\left(\frac{k}{i}\right) \mu\left(\frac{k}{j}\right). \quad \square$$

Example 3: If S^2 is the 4×4 almost Hilbert-Smith matrix, then

$$S^2 = \begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{9} & \frac{1}{16} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{36} & \frac{1}{16} \\ \frac{1}{9} & \frac{1}{36} & \frac{1}{9} & \frac{1}{144} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{144} & \frac{1}{16} \end{bmatrix}.$$

Moreover

$$d_{11} = 1 \cdot 1 \cdot \left(\frac{\mu(1)\mu(1)}{J_2(1)} + \frac{\mu(2)\mu(2)}{J_2(2)} + \frac{\mu(3)\mu(3)}{J_2(3)} + \frac{\mu(4)\mu(4)}{J_2(4)} \right) = \frac{35}{24},$$

$$d_{12} = 1 \cdot 2 \cdot \left(\frac{\mu(2)\mu(1)}{J_2(2)} + \frac{\mu(4)\mu(2)}{J_2(4)} \right) = -\frac{4}{3}, \quad d_{13} = 1 \cdot 3 \cdot \frac{\mu(3)\mu(1)}{J_2(3)} = -\frac{9}{8},$$

$$d_{14} = 1 \cdot 4 \cdot \frac{\mu(4)\mu(1)}{J_2(4)} = 0, \quad d_{22} = 2 \cdot 2 \cdot \left(\frac{\mu(1)\mu(1)}{J_2(2)} + \frac{\mu(2)\mu(2)}{J_2(4)} \right) = \frac{20}{3}, \quad d_{23} = 0,$$

$$d_{24} = 2 \cdot 4 \cdot \frac{\mu(2)\mu(1)}{J_2(4)} = -\frac{16}{3}, \quad d_{33} = 3 \cdot 3 \cdot \frac{\mu(1)\mu(1)}{J_2(3)} = \frac{81}{8}, \quad d_{34} = 0, \quad d_{44} = 4 \cdot 4 \cdot \frac{\mu(1)\mu(1)}{J_2(4)} = \frac{64}{3}.$$

Therefore, since $(S^2)^{-1}$ is symmetric, we have

$$(S^2)^{-1} = \begin{bmatrix} \frac{35}{24} & -\frac{4}{3} & -\frac{9}{8} & 0 \\ -\frac{4}{3} & \frac{20}{3} & 0 & -\frac{16}{3} \\ -\frac{9}{8} & 0 & \frac{81}{8} & 0 \\ 0 & -\frac{16}{3} & 0 & \frac{64}{3} \end{bmatrix}.$$

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