PERFECT SQUARES IN THE LUCAS NUMBERS

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1. INTRODUCTION

We consider two sequences defined by the recursion relations

$$u_0 = 0,\ u_1 = 1, u_{n+2} = au_{n+1} - bu_n, \tag{1}$$

$$v_0 = 2,\ v_1 = a,\ v_{n+2} = av_{n+1} - bv_n, \tag{2}$$

where a and b are integers which are nonzero, $D = a^2 - 4b \neq 0$. Then

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},\quad v_n = \alpha^n + \beta^n, \tag{3}$$

where α and β are distinct roots of the polynomial $f(z) = z^2 - az + b$. Each u_n is called a *Lucas number*, which is an integer. A Lucas sequence $\{u_n\}$ is called *degenerate* if the quotient of the roots of f is a root of unity and *nondegenerate* otherwise. Throughout this paper we assume that a and b are coprime.

The problem of determining all the perfect squares in a Lucas sequence has been studied by several authors: Cohn, Halton, Shorey, Tijdeman, Ribenboim, Mcdaniel, among others. In 1964, Cohn [1], [2] proved that when $a = 1$ and $b = -1$, the only squares in the sequence $\{u_n\}$ are $u_0 = 0$, $u_1 = u_2 = 1$, and $u_{12} = 144$, and the only squares in the sequence $\{v_n\}$ are $v_1 = 1$ and $v_3 = 4$. In 1969, by using the theory of elliptic curves, London and Finkelstein [5] proved that the only cubes in the Fibonacci sequence are $F_0 = 0$, $F_1 = F_2 = 1$, and $F_6 = 8$. Shorey and Tijdeman [9] proved for nondegenerate Lucas sequences that given $d \neq 0$ and $e \geq 2$, where d and e are integers, if $u_m = dU^e$ with $U \neq 0$ (U integral), then m is bounded by an effectively computable constant. In 1996, Ribenboim and Mcdaniel [8] proved that, if a and b are odd and coprime and if $D = a^2 - 4b$ is positive, then u_n is a perfect square only if $n = 0, 1, 2, 3, 6$, or 12, v_n is a perfect square only if $n = 1, 3$, or 5.

The aim of this paper is to give an elementary proof of a special case of the above result obtained by Shorey and Tijdeman [9]. Developing the argument of London and Finkelstein [5], we obtain the following results.

Proposition 1: Let $n \geq 0$ be an integer of the form $n = 4m + r$ with $0 \leq r < 4$. If u_n is a perfect square, then the rational point $(Ds^2/b^{2m}, Dst/b^{3m})$ lies on the elliptic curve $y^2 = x^3 + 4Db^r x$, where $D = a^2 - 4b$, $s^2 = |u_n|$, $t = v_n$, all of which are prime to b.

Proposition 2: Let $0 \leq r < 4$ be a fixed integer. If b is even and the group of rational points on the elliptic curve $y^2 = x^3 + 4Db^r x$ has rank zero or rank one, then u_{4m+r} is a perfect square only for finitely many $m \geq 0$.

2. PROOFS OF PROPOSITIONS 1 AND 2

Proof of Proposition 1

Let α and β be distinct roots of the polynomial $f(z) = z^2 - az + b$. Since $\alpha\beta = b$ and $D = (\alpha-\beta)^2$ we obtain, from (3), $v_n^2 - Du_n^2 = 4b^n$. Suppose that the n^{th} term u_n is a perfect square. Putting $|u_n| = s^2$ and $v_n = t$, from the equality above we have $t^2 = Ds^4 + 4b^n$. Multiplying through by D^2s^2, we see

$$(Dst)^2 = (Ds^2)^3 + 4D(Ds^2)b^n.$$

Writing $n = 4m + r$ with $0 \le r < 4$, we obtain

$$\left(\frac{Dst}{b^{3m}}\right)^2 = \left(\frac{Ds^2}{b^{2m}}\right)^3 + 4Db^r\left(\frac{Ds^2}{b^{2m}}\right).$$

Next we shall show that Ds^2/b^{2m} and Dst/b^{3m} are in lowest terms. Let p be an arbitrary prime divisor of b. Then, from (1) and (2), we have $u_n \equiv a^{n-1} \pmod{p}$ and $v_n \equiv a^n \pmod{p}$. Since a and b are coprime, $u_n \not\equiv 0 \pmod{p}$ and $v_n \not\equiv 0 \pmod{p}$; furthermore, $D = a^2 - 4b \equiv a^2 \not\equiv 0$ (mod p). We have thus completed the proof. □

Before proceeding to the proof of Proposition 2, we will need the following information.

Let c be a nonzero integer and let C be the elliptic curve given by the equation $y^2 = x^3 + cx$. We denote by Γ the additive group of rational points on C and by O the zero element of Γ.

Definition 1: For $P = (x, y) \in \Gamma$, we write $x = p/q$ in lowest terms and define the logarithmic height of P by

$$h(P) = \log\max(|P|, |q|).$$

Definition 2: For $P \in \Gamma$, the quantity

$$\hat{h}(P) = \lim_{n\to\infty}\frac{h(2^n P)}{4^n}$$

is called the *canonical height of* P.

The following two fundamental theorems on the height are well known, so the proofs are omitted (see [4] or [10]).

Theorem 1: There is a constant κ_0 that depends on the elliptic curve C, so that

$$|h(2P) - 4h(P)| \le \kappa_0 \text{ for all } P \in \Gamma. \tag{4}$$

Theorem 2 (Néron): There is a constant κ_1 that depends only on the elliptic curve C, so that for all positive integers n and for all $P \in \Gamma$ we have

$$|h(nP) - n^2\hat{h}(P)| \le \kappa_1. \tag{5}$$

Definition 3: For $P = (x, y)$ in Γ, we write $x = p/q$ in lowest terms and denote by $\lambda(P)$ the exponent of the highest power of 2 that divides the denominator q. By convention, we define $\lambda(O) = 0$.

Lemma 1: Let $P \in \Gamma$ with $P \ne (0, 0)$. If $\lambda(P) \ne 0$, then $\lambda(2P) = \lambda(P) + 2$.

Proof: We can write $P=(x,y)=(m/e^2, n/e^3)$, where m/e^2 and n/e^3 are in lowest terms with $e>0$. Then the x coordinate of $2P$ is given by

$$x(2P)=-2x+\left(\frac{3x^2+c}{2y}\right)^2=\frac{(m-ce^4)^2}{(2en)^2}.$$

Since e is even and m, n are odd, $\lambda(2P)=\lambda(P)+2$. □

Lemma 2: Let P_1 and P_2 be in Γ with $P_1\neq(0,0)$ and $P_2\neq(0,0)$. If $0\le\lambda(P_1)<\lambda(P_2)$, then $\lambda(P_1+P_2)\le\lambda(P_2)$.

Proof: If $P_1=O$, then $\lambda(P_1+P_2)=\lambda(P_2)$. So let us write $P_1=(x_1,y_1)=(m/e^2,n/e^3)$ and $P_2=(x_2,y_2)=(\overline{m}/f^2,\overline{n}/f^3)$, where m/e^2, n/e^3, $\overline{m}/f^2$, and $\overline{n}/f^3$ are in lowest terms with $e>0$ and $f>0$. Then the x coordinate of P_1+P_2 is given by

$$\begin{aligned}x(P_1+P_2)&=-x_1-x_2+\left(\frac{y_1-y_2}{x_1-x_2}\right)^2\\&=\frac{(nf^3-\overline{n}e^3)^2-(mf^2-\overline{m}e^2)^2(mf^2+\overline{m}e^2)}{e^2f^2(mf^2-\overline{m}e^2)^2}.\end{aligned}$$

Since $0\le\lambda(P_1)<\lambda(P_2)$, we can write $e=2^se'$ and $f=2^tf'$, where e' and f' are odd and s and t are integers with $0\le s<t$. Then $x(P_1+P_2)$ becomes

$$\frac{(2^{3t-3s}nf'^3-\overline{n}e'^3)^2-(2^{2t-2s}mf'^2-\overline{m}e'^2)^2(2^{2t-2s}mf'^2+\overline{m}e'^2)}{2^{2t}e'^2f'^2(2^{2t-2s}mf'^2-\overline{m}e'^2)^2}.$$

Since e', f', $\overline{m}$, and $\overline{n}$ are odd, we have $\lambda(P_1+P_2)\le 2t$. Combining this with $\lambda(P_2)=2t$, we obtain $\lambda(P_1+P_2)\le\lambda(P_2)$. □

Lemma 3: Assume that Γ has rank one, and let P be a generator for the infinite cyclic subgroup of Γ. Let t_0 denote the least positive value of the integer t such that $\lambda(tP)\neq 0$. Then, for any integer $l\ge 0$, if $2^lt_0\le n<2^{l+1}t_0$, then $\lambda(nP)\le\lambda(2^lt_0P)$.

Proof: We use strong induction on l. First we show that the result is true for $l=0$. Suppose $t_0\le n<2t_0$. Then we can write $n=t_0+r$ with $0\le r<t_0$. Since $\lambda(rP)=0$ and $\lambda(t_0P)>0$, by Lemma 1 we have $\lambda(nP)=\lambda(t_0P+rP)\le\lambda(t_0P)$.

Next we suppose that the result is true for each $l=0,1,2,\dots,k$. For any integer n satisfying $2^{k+1}t_0\le n<2^{k+2}t_0$, there exists an integer r such that $n=2^{k+1}t_0+r$ and $0\le r<2^{k+1}t_0$. The induction hypothesis gives $\lambda(rP)\le\lambda(2^kt_0P)$. By Lemma 1 we have $\lambda(2^kt_0P)<\lambda(2^{k+1}t_0P)$. Therefore, $\lambda(rP)<\lambda(2^{k+1}t_0P)$; thus, by Lemma 2 we have $\lambda(nP)=\lambda(2^{k+1}t_0P+rP)\le\lambda(2^{k+1}t_0P)$, which shows that the result is true for $l=k+1$. Hence, the result is true for every integer $l\ge 0$ and the proof is complete. □

Proof of Proposition 2

We put $R_m=(Ds^2/b^{2m}, Dst/b^{3m})$, where $s^2=|u_{4m+r}|$ and $t=v_{4m+r}$. Assume that Γ has rank zero. Then it is a finite cyclic group, and so the rational point R_m lies on the elliptic curve C only for finitely many $m\ge 0$; therefore, u_{4m+r} is a perfect square only for finitely many $m\ge 0$.

Next assume that Γ has rank one. Then $\Gamma \cong Z \oplus F$, where Z is an infinite cyclic group and F is a torsion group of order two or four (see [4] or [10]). Let $P \in \Gamma$ be a generator for Z and $Q \in \Gamma$ for F. Now suppose that the rational point R_m lies on the elliptic curve C. Then there are integers i and j such that

$$R_m = iP + jQ. \tag{6}$$

Since $4Q = O$, where O is the zero element of Γ, we obtain

$$4R_m = 4iP. \tag{7}$$

The essential tool for the proof is the logarithmic height. Since $h(4iP) = h(-4iP)$, we can assume $i > 0$ without loss of generality. Let k_0 be the least positive value of the integer k such that $\lambda(kP) \neq 0$. Then there is an integer $l \geq 0$ such that $2^l k_0 \leq 4i < 2^{l+1} k_0$. From Lemmas 1 and 3, we find $\lambda(4iP) \leq \lambda(2^l k_0 P) = \lambda(k_0 P) + 2l$. Since $\lambda(4iP) = \lambda(4R_m) > 2m$, putting $\lambda_0 = \lambda(k_0 P)$, we obtain $2l > \lambda(4iP) - \lambda_0 > 2m - \lambda_0$. Hence, $4i \geq 2^l k_0 > 2^{m-\lambda_0/2}$.

Now, Theorem 2 tells us that there is a constant K_1 depending only on the elliptic curve C, so that

$$h(4iP) \geq (4i)^2 \hat{h}(P) - K_1 > 2^{2m-\lambda_0} \hat{h}(P) - K_1. \tag{8}$$

Next we estimate for $h(4R_m)$. Let α and β be distinct roots of the polynomial $f(z) = z^2 - az + b$. Putting $\gamma = \max(|\alpha|, |\beta|) \geq 1$, we find

$$|b^{2m}| = |\alpha\beta|^{2m} \leq \gamma^{4m},$$

$$\begin{aligned} |Ds^2| = |Du_{4m+r}| &= |\alpha - \beta||\alpha^{4m+r} - \beta^{4m+r}| \\ &\leq (|\alpha| + |\beta|)(|\alpha|^{4m+r} + |\beta|^{4m+r}) \leq 4\gamma^{4m+4} \end{aligned}$$

Therefore, $h(R_m) \leq \log 4\gamma^{4(m+1)} = 4(m+1)\log\gamma + 2\log 2$. Hence, by Theorem 1,

$$h(4R_m) \leq 16h(R_m) + 5K_0 \leq 64(m+1)\log\gamma + 32\log 2 + 5K_0, \tag{9}$$

where K_0 is a constant depending only on the elliptic curve C.

It follows that, if the rational point R_m lies on the elliptic curve C, then m satisfies the following inequality:

$$64(m+1)\log\gamma + 32\log 2 + 5K_0 > 2^{2m-\lambda_0} \hat{h}(P) - K_1. \tag{10}$$

However, there exists a constant $N > 0$ such that inequality (10) is false for every $m \geq N$, so the rational point R_m is not found on C for every $m \geq N$. We conclude from Proposition 1 that u_{4m+r} is not a perfect square for every $m \geq N$. We have thus completed the proof. □

3. APPLICATIONS

Following Silverman and Tate [10], we describe how to compute the rank r of the group Γ of rational points on the elliptic curve C: $y^2 = x^3 + cx$ with integral coefficients. Let $\mathbb{Q}^*$ denote the multiplicative group of nonzero rational numbers, and let $\mathbb{Q}^{*2} = \{u^2 : u \in \mathbb{Q}^*\}$. Now consider the map $\varphi : \Gamma \to \mathbb{Q}^* / \mathbb{Q}^{*2}$ defined by the rule:

$$\begin{aligned} &\varphi(O) = 1 \quad (\mathrm{mod}\, \mathbb{Q}^{*2}) \\ &\varphi(0,0) = c \quad (\mathrm{mod}\, \mathbb{Q}^{*2}) \\ &\varphi(x,y) = x \quad (\mathrm{mod}\, \mathbb{Q}^{*2}) \text{ if } x \neq 0. \end{aligned}$$

On the other hand, let $\overline{\Gamma}$ denote the group of rational points on the elliptic curve $\overline{C}: y^2 = x^3 - 4cx$. Using the analogous map $\overline{\varphi}: \overline{\Gamma} \to \mathbb{Q}^* / \mathbb{Q}^{*2}$, we obtain the formula for the rank of Γ:

$$2^r = \frac{\#\varphi(\Gamma) \cdot \#\overline{\varphi}(\overline{\Gamma})}{4}, \tag{11}$$

where $\#\varphi(\Gamma)$ and $\#\overline{\varphi}(\overline{\Gamma})$ denote the order of $\varphi(\Gamma)$ and the order of $\overline{\varphi}(\overline{\Gamma})$, respectively.

Next we describe how to determine the order of $\varphi(\Gamma)$. It is obvious from the rule of the map φ that $\{1, c(\text{mod}\,\mathbb{Q}^{*2})\} \subset \varphi(\Gamma)$.

Now, for $P = (x, y) \in \Gamma$ with $y \neq 0$, the coordinates x and y are written in the form

$$x = \frac{c_1 M^2}{e^2}, \quad y = \frac{c_1 MN}{e^3}$$

in lowest terms with $M \neq 0$ and $e > 0$, where c_1 is an integral divisor of c, so that $c = c_1 c_2$. Here M, e, and N must satisfy the equation

$$N^2 = c_1 M^4 + c_2 e^4, \tag{12}$$

and also the conditions

$$\gcd(M, e) = \gcd(N, e) = \gcd(c_1, e) = 1,$$
$$\gcd(c_2, M) = \gcd(M, N) = 1.$$

Hence, for a factorization $c = c_1 c_2$, if the equation $N^2 = c_1 M^4 + c_2 e^4$ has a solution (M, e, N) with $M \neq 0$ that satisfies the side conditions above, then c_1 (mod $\mathbb{Q}^{*2}$) is in $\varphi(\Gamma)$, otherwise it is not.

Proposition 3: Let p be a prime and let C be the elliptic curve $y^2 = x^3 - 4px$. If $p \equiv 3$ (mod 4), then the group Γ of rational points on C has rank zero or rank one.

Proof: Since $c = -4p$, the possibilities for c_1 are $c_1 = \pm 1, \pm 2, \pm 4, \pm p \pm 2p, \pm 4p$. So we see that $\varphi(\Gamma) \subset \{\pm 1, \pm 2, \pm p, \pm 2p \text{ (mod}\,\mathbb{Q}^{*2})\}$. We shall show first that $-1 \notin \Gamma$. Let us consider the equation

$$N^2 = -M^4 + 4pe^4. \tag{13}$$

This implies the congruence $N^2 \equiv -M^4$ (mod p). Since $p \equiv 3$ (mod 4), we have $(-1/p) = -1$, where $(-1/p)$ is the Legendre symbol of -1 for p; hence, the congruence above has no solutions with $M \not\equiv 0$ (mod p). So equation (13) has no solutions in integers with $\gcd(M, N) = 1$. Similarly, the equation $N^2 = -4M^4 + pe^4$ has no solutions in integers with $\gcd(M, N) = 1$. Therefore, $-1 \notin \varphi(\Gamma)$, and hence $\#\varphi(\Gamma) = 2$ or $\#\varphi(\Gamma) = 4$.

On the other hand, let $\overline{C}$ be the elliptic curve $y^2 = x^3 + 16px$, and let $\overline{\Gamma}$ denote the group of rational points on $\overline{C}$. Since $\overline{c} = 16p$, we have $\overline{\varphi}(\overline{\Gamma}) \subset \{1, 2, p, 2p \text{ (mod}\,\mathbb{Q}^{*2})\}$. We shall show by contradiction that $2 \notin \overline{\varphi}(\overline{\Gamma})$. Let us consider the equation

$$N^2 = 2M^4 + 8pe^4. \tag{14}$$

Suppose equation (14) has a solution in integers with $M \neq 0$ and $\gcd(M, N) = 1$. Then N is even. Putting $N = 2N_1$, we have $2N_1^2 = M^4 + 4pe^4$, showing that M is even, contrary to the hypothesis that M and N are coprime. Hence, equation (14) has no solutions in integers with $\gcd(M, N) = 1$. Similarly, the equation $N^2 = 8M^4 + 2pe^4$ has no solutions in integers with $\gcd(N, e) = 1$. Thus, $2 \notin \overline{\varphi}(\overline{\Gamma})$, and so $\#\overline{\varphi}(\overline{\Gamma}) = 2$. By formula (11), we find

$$2^r = \frac{\#\varphi(\Gamma)\cdot\#\overline{\varphi}(\overline{\Gamma})}{4} = 1 \text{ or } 2.$$

Therefore, Γ has rank zero or rank one. □

Proposition 4: Let p, q be primes and let C be the elliptic curve $y^2 = x^3 - 4pqx$. If $p \equiv 5 \pmod 8$, $q \equiv 3 \pmod 8$, and $(p/q) = -1$, then the group Γ of rational points on C has rank zero or rank one.

Proof: Since $c = -4pq$, we have $\varphi(\Gamma) \subset \{\pm 1, \pm 2, \pm p, \pm q, \pm 2p, \pm 2q, \pm pq, \pm 2pq \pmod{\mathbb{Q}^{*2}}\}$. We shall show, for instance, that $-2p \notin \Gamma$. The hypotheses give

$$\left(\frac{-1}{p}\right) = 1,\ \left(\frac{2}{p}\right) = \left(\frac{-1}{q}\right) = \left(\frac{2}{q}\right) = -1,\ \left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = -1.$$

Hence, the congruence $N^2 \equiv -2pM^4 \pmod q$ has no solutions with $M \not\equiv 0 \pmod q$ because $(-2p/q) = (-1/q)(2/q)(p/q) = -1$, so $N^2 = -2pM^4 + 2qe^4$ has no solutions in integers with $\gcd(M, N) = 1$. Therefore, $-2p \notin \Gamma$. By using the same argument, we can show that $\varphi(\Gamma)$ does not have any elements of $\{-1, \pm 2, \pm p, \pm q, -2p, pq, 2pq\}$. Thus, we obtain $\#\varphi(\Gamma) \leq 4$.

On the other hand, let $\overline{C}$ be the elliptic curve $y^2 = x^3 + 16pqx$, and let $\overline{\Gamma}$ denote the group of rational points on $\overline{C}$. Since $\overline{c} = 16pq$, we have $\overline{\alpha}(\overline{\Gamma}) \subset \{1, 2, p, q, 2p, 2q, pq, 2pq \pmod{\mathbb{Q}^{*2}}\}$. By using an argument similar to the one above, we can show that $p \notin \overline{\varphi}(\overline{\Gamma})$ and $q \notin \overline{\varphi}(\overline{\Gamma})$. Furthermore, by using an argument similar to the one we gave in the proof of Proposition 3, we can show that $\overline{\varphi}(\overline{\Gamma})$ does not have any elements of $\{2, 2p, 2q, 2pq\}$. Thus, we obtain $\#\varphi(\Gamma) = 2$. Therefore, by formula (11), we find $2^r \leq 2$. In conclusion, Γ has rank zero or rank one. □

In addition, the following proposition holds. The proof is completely analogous to that of Proposition 4.

Proposition 5: Let p, q be primes and let C be the elliptic curve $y^2 = x^3 - 4pqx$. If $p \equiv 1 \pmod 8$, $q \equiv 7 \pmod 8$, and $(p/q) = -1$, then the group Γ of rational points on C has rank zero or rank one.

Now let us consider the Lucas sequence determined by $u_0 = 0$, $u_1 = 1$, $u_{n+2} = au_{n+1} - bu_n$, where a and b are coprime integers that are nonzero, $D = a^2 - 4b \neq 0$. Assume that b is even. If $D = -p < 0$, where p is a prime, then $p \equiv 3 \pmod 4$. If $D = -pq < 0$, where p and q are primes, then $(p, q) \equiv (3, 5) \pmod 8$ or $(p, q) \equiv (1, 7) \pmod 8$. Hence, the following three corollaries hold.

Corollary 1: Assume b is even and $D = -p < 0$, where p is a prime. Then there are only finitely many perfect squares in the subsequence $\{u_{4m}\}$.

Corollary 2: Assume b is even and $D = -pq < 0$, where p and q are primes with $(p/q) = -1$. Then there are only finitely many perfect squares in the subsequence $\{u_{4m}\}$.

Corollary 3: Assume b is of the form $b = (2d)^4$ for some integer d. If $D = -p$, where p is a prime, or if $D = -qr < 0$, where q and r are primes with $(q/r) = -1$, then there are only finitely many perfect squares in the sequence $\{u_n\}$.

Proof: Suppose that the n^{th} term, u_n, is a perfect square. As mentioned above, we have $t^2 = Ds^4 + 4(2d)^{4n}$, where $s^2 = |u_n|$ and $t = v_n$. This implies

$$\left\{\frac{Dst}{(2d)^{3n}}\right\}^2 = \left\{\frac{Ds^2}{(2d)^{2n}}\right\}^3 + 4D\left\{\frac{Ds^2}{(2d)^{2n}}\right\}.$$

From Propositions 3, 4, and 5, we obtain that the elliptic curve $y^2 = x^3 + 4Dx$ has rank zero or rank one. It follows that u_n is a perfect square only for finitely many $n \geq 0$. □

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