Stern's Diatomic Array Applied to Fibonacci Representations

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1. Stern's Diatomic Array

Each row of Pascal's triangle is formed by addition of adjacent elements of the preceding row, producing binomial coefficients and counting combinations. Each row of Stern's diatomic array is formed by addition of adjacent elements of the preceding row, but interspersed with elements of the preceding row. In this case, the rows of the table will be shown to count certain Fibonacci representations.

Starting with 1 and 1, form a table in which each line is formed by copying the preceding line, and inserting the sum of consecutive elements: 1, 1; 1, 1+1, 1; 1, 1+2, 2, 2+1, 1; . . . .

The body of the table contains Stern's diatomic array, sequence A049456 in [10]. Actually, this array has been called Stern's diatomic series in the literature [9], [11], but it should have been called the Eisenstein-Stern diatomic series by earlier authors because Stern's introduction refers back to Eisenstein.

\[
\begin{array}{cccccccccccccccc}
1 & & & & & & & & & & & & & & 1 \\
1 & 2 & & & & & & & & & & & & & & 1 \\
1 & 3 & 5 & & & & & & & & & & & & & & 1 \\
1 & 4 & 3 & 8 & & & & & & & & & & & & & & 1 \\
1 & 5 & 4 & 7 & 3 & 8 & 5 & & & & & & & & & & 1 \\
\end{array}
\]

If \(a_{n,k}\) is the \(k^{th}\) term in the \(n^{th}\) line, \(k = 1, 2, \ldots, n = 0, 1, 2, \ldots\),

\[
a_{n,2m} = a_{n-1,m} + a_{n-1,m+1} \quad \text{and} \quad a_{n,2m-1} = a_{n-1,m}.
\]  \(1.1\)

Lehmer [9] reports that Stern took the initial line 1, 1 as the zero\(^{th}\) line and proved, among others, the following properties:

1. The number of terms in the \(n^{th}\) line is \(2^n + 1\), and their sum is \(3^n + 1\).
2. The table is symmetric; in the \(n^{th}\) line the \(k^{th}\) term equals the \((2^n + 2 - k)^{th}\) term.
3. Terms appearing in the \(n^{th}\) line as sums of their two adjacent terms are called dyads. There are \(2^n - 1\) dyads and \(2^{n-1} + 1\) non-dyads on the \(n^{th}\) line. The dyads occupy positions of even number \(k\) (called rank) on the line.
4. Two consecutive terms, \(a\) and \(b\), have no common factor.
5. Every ordered pair \((a, b)\) occurs exactly once as consecutive terms in some line of the table.
6. If \(a\) and \(b\) are relatively prime, the pair of consecutive terms \((a, b)\) appears in the line whose number is one less than the sum of the quotients appearing in the expansion of \(a/b\) in a regular continued fraction.

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Lehmer [9] then uses the quotients of the continued fraction expansion of \( r_1/r_2 \) to place the consecutive terms \( r_1 \) and \( r_2 \) into the table by computing both the line number and the rank of \( r_1 \). Further, he shows that the largest dyads in the \( n \)th line, \( n \geq 2 \), have the value \( F_{n+2} \), the \((n + 2)\)nd Fibonacci number. Lehmer’s results for the line number and rank are summarized in Theorem 1.1.

**Theorem 1.1:** If consecutive terms \( r_1 \) and \( r_2 \) occur on the \( n \)th line of Stern’s diatomic array and if the continued fraction for \( r_1/r_2 \) is \( [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}] \), then

\[
    n = q_1 + q_2 + q_3 + \cdots + q_{m-2} + r_{m-1} - 1,
\]

and if \( m \) is odd (even), \( r_1 \) is the left (right) neighbor of \( r_2 \) in the first (second) half of line \( n \). If \( m \) is odd, the position number \( k \) (rank) of \( r_1 \) in the first half of line \( n \) is

\[
    k = 2^{q_1} + 2^{q_2} + \cdots + 2^{q_{m-2}} - 2^{q_1 + q_2 + \cdots + q_{m-2}} + \cdots - 2^{q_1 + q_2} + 2^n.
\]

More recently, Calkin and Wilf [6] use Stern’s diatomic array to explicitly describe a sequence \( b(n) \) (sequence A002487 in [10]) such that every positive rational appears exactly once as \( b(n)/b(n+1) \),

\[
    \{b(n)\} = \{1, 1, 2, 1, 3, 2, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, \ldots\}.
\]

It is shown in [6] that \( b(n) \) counts the number of hyperbinary representations of the integer \( n, n \geq 1 \); that is, the number of ways of writing \( n \) as a sum of powers of 2, each power being used at most twice, \( b(0) = 1 \). Here, we apply Stern’s diatomic array to counting Fibonacci representations.

2. FIBONACCI REPRESENTATIONS

Let \( R(N) \) denote the number of Fibonacci representations [4] of the positive integer \( N \); that is, the number of representations of \( N \) as sums of distinct Fibonacci numbers \( F_k \), (or as a single Fibonacci number \( F_k \)), \( k \geq 2 \), written in descending order. We define \( R(0) = 1 \). The Zeckendorf representation of \( N \), denoted Zeck \( N \), is the unique representation of \( N \) using only non-consecutive Fibonacci numbers \( F_k \), \( k \geq 2 \). The largest Fibonacci number contained in \( N \) will be listed first in Zeck \( N \). Whenever \( R(N) \) is prime, Zeck \( N \) uses only Fibonacci numbers whose subscripts have the same parity [3], [5]. For that reason, we are interested in integers \( N \) whose Zeckendorf representation uses only even-subscripted Fibonacci numbers; we call such \( N \) an even-Zeck integer, denoted \( \hat{N} \), sequence A054204 in [10]. The \( j \)th even-Zeck integer \( \hat{N} = \hat{N}(j) \) can be written immediately when \( j \) is known.

We list early values \( R(\hat{N}) \) for consecutive even-Zeck integers \( \hat{N} \), augmented with \( R(0) = 1 \), sequence A002487 in [10]:

\[
\begin{align*}
    j & \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad \ldots \\
    \hat{N}(j) & \quad 0 \quad 1 \quad 3 \quad 4 \quad 8 \quad 9 \quad 11 \quad 12 \quad 21 \quad 22 \quad 24 \quad 25 \quad 29 \quad 30 \quad 32 \quad 33 \quad \ldots \\
    R(\hat{N}(j)) & \quad 1 \quad 1 \quad 2 \quad 1 \quad 3 \quad 2 \quad 3 \quad 1 \quad 4 \quad 3 \quad 5 \quad 2 \quad 5 \quad 3 \quad 4 \quad 1 \quad \ldots
\end{align*}
\]
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The even-Zeck integers $\tilde{N}(j)$ are enumerated below for $j = 1, 2, \ldots$; we define $\tilde{N}(0) = 0$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>binary</th>
<th>powers of 2</th>
<th>Zeck $\tilde{N}(j)$</th>
<th>$\tilde{N}(j)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$2^0$</td>
<td>$F_2$</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>$2^1$</td>
<td>$F_4$</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
<td>$2^1 + 2^0$</td>
<td>$F_4 + F_2$</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>$2^2$</td>
<td>$F_8$</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>101</td>
<td>$2^2 + 2^0$</td>
<td>$F_8 + F_2$</td>
<td>9</td>
</tr>
<tr>
<td>6</td>
<td>110</td>
<td>$2^2 + 2^1$</td>
<td>$F_8 + F_4$</td>
<td>11</td>
</tr>
<tr>
<td>7</td>
<td>111</td>
<td>$2^2 + 2^1 + 2^0$</td>
<td>$F_8 + F_4 + F_2$</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>1000</td>
<td>$2^3$</td>
<td>$F_8$</td>
<td>21</td>
</tr>
<tr>
<td>9</td>
<td>1001</td>
<td>$2^3 + 2^0$</td>
<td>$F_8 + F_2$</td>
<td>22</td>
</tr>
<tr>
<td>10</td>
<td>1010</td>
<td>$2^3 + 2^1$</td>
<td>$F_8 + F_4$</td>
<td>24</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Lemma 2.1: If $j$ is represented as the sum of distinct powers of 2 in descending order, $j = 2^r + 2^s + \cdots + 2^w$, $r > s > w$, then the $j$th even-Zeck integer $\tilde{N} = \tilde{N}(j)$ is given by Zeck $\tilde{N}(j) = F_{2(r+1)} + F_{2(s+1)} + \cdots + F_{2(w+1)}$. In short, replace the binary representation of $j$ in the powers $2^p$, $p = 0, 1, \ldots$, by $F_{2(p+1)}$ to find $\tilde{N} = \tilde{N}(j)$.

Proof: The short table displays Lemma 2.1 for $j = 1, 2, \ldots, 10$. The next even-Zeck integer $\tilde{N}(j+1)$ will be formed from the binary representation of $(j+1)$.

Lemma 2.2. (i) If Zeck $\tilde{N} = \tilde{N}(j)$, $j \geq 2$, has $F_2$ for its smallest term, then $\tilde{N} - 1 = \tilde{N}(j-1)$, but $\tilde{N} + 1$ is not an even-Zeck integer.

(ii) If Zeck $\tilde{N} = \tilde{N}(j)$, $j \geq 2$, has $F_{2c}$, $c \geq 2$, for its smallest term, then $\tilde{N} + 1 = \tilde{N}(j+1)$, but $\tilde{N} - 1$ is not an even-Zeck integer.

(iii) The even-Zeck integer $\tilde{N}^*$ preceding $\tilde{N} = \tilde{N}(j)$, $j \geq 2$, with $F_{2c}$, $c \geq 1$, for its smallest term, is $\tilde{N}(j-1) = \tilde{N}^* = \tilde{N} - F_{2c-2} - 1$.

Proof: Let $c = 1$, and take $\tilde{N}(j) = F_{2n} + \cdots + F_{2p} + F_2$, $p \geq 2$, $n \geq 3$. Then $\tilde{N} - 1 = \tilde{N}(j-1)$, but $\tilde{N} + 1 = F_{2n} + \cdots + F_{2p} + F_3$ is not an even-Zeck integer, illustrating (i). Further, $\tilde{N}(j-1) = \tilde{N} - F_0 - 1 = \tilde{N} - F_{2c-2} - 1$, $c = 1$, satisfying (iii).

Let $\tilde{N}(j) = F_{2n} + \cdots + F_{2c+2p} + F_{2c}$, $c \geq 2$, $p \geq 1$, $n \geq 3$. Then $\tilde{N}(j) + 1 = F_{2n} + \cdots + F_{2c+2} + F_{2c} + F_2 = \tilde{N}(j+1)$, but $\tilde{N}(j) - 1 = F_{2n} + \cdots + F_{2c+2p} + F_{2c} - 1 = F_{2n} + \cdots + F_{2c+2p} + (F_{2c-1} + \cdots + F_7 + F_5 + F_3)$, not an even-Zeck integer, as in (ii). Part (iii) follows from

$$\tilde{N}(j) - F_{2c-2} - 1 = F_{2n} + \cdots + F_{2c+2p} + F_{2c} - F_{2c-2} = 1$$
$$= F_{2n} + \cdots + F_{2c+2} + (F_{2c-1} - 1)$$
$$= F_{2n} + \cdots + F_{2c+2p} + (F_{2c-2} + \cdots + F_6 + F_4 + F_2) = \tilde{N}(j-1) = \tilde{N}^*.$$
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Cut the list from $R(\tilde{N})$ given earlier in (2.1) at the boundary 1's to form rows

$$(1, 1), \ (1, 2, 1), \ (1, 3, 2, 3, 1), \ (1, 4, 3, 5, 2, 5, 3, 4, 1), \ \ldots,$$

where we keep the leftmost 1 for symmetry. Each row, after the first, includes the list of $R(\tilde{N})$ for the preceding row, interspersed with sums of successive pairs of adjacent terms from the preceding row:

$$(1, 1), \ (1, 1+1, 1), \ (1, 2+1, 2, 2+1, 1), \ (1, 1+3, 3, 3+2, 2, 2+3, 3, 3+1, 1), \ \ldots.$$

We recognize the first four lines of Stern's Diatomic array. Our \(n\)th row, \(n, (n-1), \ldots\), contains 1 followed by the number of Fibonacci representations \(R(\tilde{N})\) for consecutive even-Zeck integers \(\tilde{N}, F_{2n} \leq \tilde{N} \leq F_{2n+1} - 1\) where \(R(F_{2n}) = n, n \geq 1\). We next prove that this array is indeed the same as Stern's diatomic array. Lemma 2.3, which allows us to shift subscripts, was Hoggatt's conjecture and was proved by Klarner [8, Thm. 4]. Lemma 2.4 is part of Lemma 11 from [4].

Lemma 2.3: If sequence \(\{b_n\}\) satisfies the Fibonacci recurrence \(b_{n+2} = b_{n+1} + b_n\), then \(R(b_k - 1) = R(b_{k+1} - 1)\) for \(k\) sufficiently large.

Lemma 2.4: Let \(N\) be an integer whose Zeckendorf representation has \(F_{2c}, c \geq 2\), as its smallest term. Then \(R(N) = R(N - 1) + R(N + 1)\).

Theorem 2.1: Let the \(n\)th row of an array list the number of Fibonacci representations \(R(\tilde{N})\) for consecutive even-Zeck Integers \(\tilde{N}, F_{2n} \leq \tilde{N} \leq F_{2n+1} - 1\). Let \(b_{n,k}\) denote the \(k\)th term of the \(n\)th row, \(n = 1, 2, 3, \ldots\), where \(b_{n,1} = 1\), and \(b_{n,k} = R(\tilde{N}(j, n, k))\) for \(j, n, k \geq 2n - 1, k = 2, 3, \ldots, 2n - 1 + 1\). Then \(b_{n,1} = a_{n-1}k\), the \(k\)th term in the \((n-1)\)th line in Stern's diatomic array, \(n = 1, 2, \ldots\), and \(k = 1, 2, \ldots, 2n - 1 + 1\). We prove that the two arrays have the same rule of formation by establishing

\[
b_{n,2m} = b_{n-1,m} + b_{n-1,m+1} \text{ and } b_{n,2m-1} = b_{n-1,m}, \quad n \geq 2. \tag{2.2}
\]

(a) We first prove \(b_{n,2m} = b_{n-1,m} + b_{n-1,m+1}\) for \(n \geq 2, m = 1, \ldots, 2n - 2\). The case \(m = 1\) is satisfied because \(b_{n-1,1} = 1\) by definition, and \(b_{n-1,2} = R(\tilde{N}(2n - 2)) = R(F_{2n-1}) = n - 1\) from above. For \(m = 2, \ldots, 2n - 2\), \(\tilde{N}(j) \equiv \tilde{N}(j, 2m) = F_{2m} + \cdots + F_{2c+2p} + F_{2c}, c \geq 2, 1 \leq p \leq n - 3\), when \(n \geq 3\) because \(c = 1(F_2)\) is not present for even \(k \leq 2m\) in row \(n\). Also, \(n \geq c + p\); hence, \(1 \leq p \leq n - c \leq n - 2\) for \(n \geq 3\) if \(n = 2, p = 0\) and \(c = 2\). From Lemma 2.2, \(\tilde{N}^* = \tilde{N}(j - 1), \tilde{N} = \tilde{N}(j) \equiv \tilde{N}(j, 2m),\) and \(\tilde{N} + 1 = \tilde{N}(j + 1)\) are consecutive even-Zeck integers. Hence \(b_{n,2m - 1} = R(\tilde{N}^*), b_{n,2m} = R(\tilde{N}), b_{n,2m - 1} = R(\tilde{N} + 1)\) are consecutive entries in the \(n\)th row. Since \(\tilde{N}(j - 1)\) and \(\tilde{N}(j + 1)\) are each a term in some Fibonacci sequence, apply Lemma 2.3 to shift subscripts down 2 in the expressions for \(R(\tilde{N}(j + 1))\) and \(R(\tilde{N}(j - 1))\).
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\[ R(\tilde{N}(j + 1)) = R(\tilde{N} + 1) = R(F_{2n} + \cdots + F_{2c+2p} + F_{2c} + (F_3 - 1)) \]
\[ = R((F_{2n-2} + \cdots + F_{2c+2p-2} + F_{2c-2} + F_1) - 1) \]
\[ = R(F_{2(n-1)} + \cdots + F_{2c+2p-2} + F_{2c-2}) = R(\tilde{M}), \quad (2.3) \]

which is in the \((n - 1)^{st}\) row, and \(\tilde{M}\), the argument given above, is an even-Zeck integer. The binary representation of \(\tilde{M} = F_{2n-2} + \cdots + F_{2c+2p-2} + F_{2c-2}, \ c \geq 2, \) is obtained from the binary representation of \(\tilde{N}\) by a right-shift by one position (see Lemma 2.1). Because \(\tilde{N} \equiv \tilde{N}(j_{n,2m})\), one therefore finds \(\tilde{M} \equiv \tilde{N}(j_{n-1,m+1})\); hence by definition \(R(\tilde{M}) = b_{n-1,m+1}\). From Lemma 2.2 (iii) (with \(c \to c - 1 \geq 1\)), and Lemma 2.3,

\[ R(\tilde{N}(j - 1)) = R(\tilde{N}^*) = R(\tilde{N} - F_{2c-2} - 1) \]
\[ = R(F_{2n-2} + \cdots + F_{2c+2p-2} + F_{2c-2} - F_{2c-4} - 1) = R(\tilde{M}^*) \quad (2.4) \]

where \(\tilde{M}^*\), defined as the argument of the last \(R\), is the even-Zeck preceding \(\tilde{M}\). Hence, \(\tilde{M}^* = \tilde{N}(j_{n-1,m})\), and by definition, \(R(\tilde{M}^*) = b_{n-1,m}\). What we have to prove now is \(R(\tilde{N}) = R(\tilde{M}^*) + R(\tilde{M})\). For this we want to use Lemma 2.4 with \(N \to \tilde{N}\). We know already that \(R(\tilde{N} + 1) = R(\tilde{M})\) but \(\tilde{N} - 1\) is not an even-Zeck integer for \(c \geq 2\). However, we now show that \(R(\tilde{N} - 1) = R(\tilde{M}^*)\).

\[ R(\tilde{N} - 1) = R(F_{2n} + \cdots + F_{2c+2p} + F_{2c} - 1) \]
\[ = R(F_{2n-2c+2} + \cdots + F_{2p+2} + F_2 - 1) = R(\tilde{K}) \]

by shifting subscripts down \((2c - 2)\). Recalculate \(R(\tilde{N}^*)\) as

\[ R(\tilde{N}(j - 1)) = R(F_{2n} + \cdots + F_{2c+2p} + F_{2c} - F_{2c-2} - 1) \]
\[ = R((F_{2n} + \cdots + F_{2c+2p} + F_{2c-1} - 1) \]
\[ = R(F_{2n-2c+2} + \cdots + F_{2p+2} + F_1 - 1) = R(\tilde{K}), \]

by shifting subscripts down \((2c - 2)\). Thus,

\[ R(\tilde{N} - 1) = R(\tilde{N}(j - 1)) = R(\tilde{M}^*). \quad (2.5) \]
Therefore, \( b_{n,2m} = b_{n-1,m} + b_{n-1,m+1} \), and part (a) of the proof is finished.

(b) We prove \( b_{n,2m-1} = b_{n-1,m} \) for \( n \geq 2, m = 1, \ldots, 2^n - 2 \). For \( m = 1, b_{n,1} = 1 = b_{n-1,1} \) by definition. If \( m = 2, \ldots, 2^n - 2, b_{n,2m-1} = R(\tilde{N}(j_{n,2m-1})) = R(\tilde{N}^*) \) if we use the same notation as in part (a) of the proof. There we have already shown \( R(\tilde{N}^*) = R(\tilde{M}^*) = b_{n-1,m} \), which finishes part (b) of the proof. Together with the input \( k_{1,1} = 1 = k_{1,2} \) we have shown that \( b_{n,k} = a_{n-1,k}, n = 1, 2, \ldots, \) and \( k = 1, 2, \ldots, 2^{n-1} + 1. \)

**Corollary 2.1.1:** If \( \tilde{N} \) is an even-Zeck integer such that Zeck \( \tilde{N} \) ends in \( F_{2c}, c \geq 2 \), and if \( \tilde{N}^* \) is the preceding even-Zeck integer, then \( R(\tilde{N}^*) = R(\tilde{N} - 1) \). Also, \( R(\tilde{N}(j-1)) = R(\tilde{N}(j) - 1) \) with \( \tilde{N} = \tilde{N}(j) \).

**Proof:** See equations (2.4) and (2.5). \( \square \)

**Theorem 2.2:** Let \( \tilde{N} = \tilde{N}(j) \) be the \( j \)-th even-Zeck integer, \( j = 0, 1, 2, \ldots \), with \( \tilde{N}(0) = 0 \). If \( R(\tilde{N}) = b_{n,k} \) with \( n, k \) defined in Theorem 2.1, then \( \tilde{N} = \tilde{N}(j_{n,k}) \) with \( j_{n,k} = 2^{n-1} + k - 2, k = 1, 2, \ldots, 2^{n-1} + 1, \) for \( n = 1, 2, \ldots, \) \( \tilde{N}(j), j \geq 1, \) is obtained by replacing powers \( 2^p \) in the dual representation of \( j_{n,k} \) by \( F_{2^{(p+1)}} \); if \( j = 0 \), then \( \tilde{N} = 0 \). Alternately, \( \tilde{N} = F_{2n} + \tilde{K}(k-2) \), where \( \tilde{K}(k-2) \) is the \( (k-2)^{nd} \) even-Zeck integer.

**Proof:** Apply Lemma 2.1 to Theorem 2.1. \( \square \)

To illustrate Theorem 2.2, \( R(\tilde{N}) = 7 = b_{6,8} \) appears as the 8th term in the 5th row;
\( n - 2 = 8 - 2 = 6 = 2^2 + 2^1 \), yielding \( \tilde{N} = F_{2-8} + F_{2(2+1)} + F_{2(1+1)} = F_{10} + F_6 + F_4 = 66 \), and
\( R(66) = 7 \). The earlier \( R(\tilde{N}) = 7 = b_{6,4} \) in that row occurs for \( \tilde{N} = F_{10} + F_4 = 58 \).

Since the \( n \)-th row of the array for \( R(\tilde{N}) \) is the \( (n - 1)^{st} \) line of Stern’s diatomic array, several properties of Fibonacci representations of even-Zeck integers \( \tilde{N} \) correspond directly to properties given for elements of Stern’s diatomic array from Section 1.

1. There are \( 2^{n-1} \) even-Zeck integers \( \tilde{N} \) in the interval \( F_{2n} \leq \tilde{N} \leq F_{2n+1} - 1 \). There are \( 2^{n-1} + 1 \) terms \( R(\tilde{N}) \) in the \( n \)-th row, whose sum is \( 3^{n-1} + 1 \).
2. The table of \( R(\tilde{N}) \) values is symmetric; in the \( n \)-th row, the \( k \)-th term equals the \( \left(2^{n-1} + 2 - k\right)^{th} \) term. Compare with \( R(F_{2n} + M) = R(F_{2n+1} - 2 - M), 0 \leq M \leq F_{2n-1}, n \geq 2, \) formed from Theorem 1 of [4] by replacing \( n \) with \( 2n \).
3. Dyads \( R(\tilde{N}) \) correspond to Zeck \( \tilde{N} \) ending in \( F_{2c}, c \geq 2 \); excepting the first column, non-dyads \( R(\tilde{N}) \) have Zeck \( \tilde{N} \) ending in \( F_2 = 1 \). The dyads have even term numbers.
4. For even-Zeck \( \tilde{N} \), consecutive values for \( R(\tilde{N}) \) are relatively prime. Consecutive values for even-Zeck integers \( \tilde{N} \) appear in relatively prime pairs, \( (3,4), (8,9), (11,12), (21,22), (24,25), \ldots \)

The largest value \( [2] \) for \( R(\tilde{N}) \) in row \( n \) is \( F_{n+1} \), corresponding to \( F_{n+2} \) as the largest dyad in the \( n \)-th line as given by Lehmer [9]. Notice that Lemma 2.3 appears in the table as the columns of constants, and the central term in each row is 2. Properties 5 and 6 are explored in the next section.

3. Stern’s Diatomic Array Applied to Fibonacci Representations

We can find many even-Zeck integers \( \tilde{N} \) having a specified value for \( R(\tilde{N}) \) by applying Theorem 1.1. According to Lehmer [9], Stern gives Euler’s \( \Phi(m) \) as the number of times that an element \( m \) appears in the \( (m - 1)^{st} \) and all succeeding lines of Stern’s diatomic array; this,
Stern’s Diatomic Array Applied to Fibonacci Representations

of course, is our $m^{th}$ row, where values for $R(\tilde{N})$ are the elements, and Euler’s $\Phi(m)$ is the number of integers not exceeding $m$ and prime to $m$. We express $R(\tilde{N})$ as the sum of a pair of relatively prime integers $r_1$ and $r_2$, and then use the Euclidean algorithm to write quotients used in the continued fraction for $r_1/r_2$. The row and column numbers for $R(\tilde{N}) = b_{n,k}$, as well as the Zeckendorf representation of $\tilde{N}$, can be written from those same quotients.

**Theorem 3.1:** Let $R(\tilde{N}) = b_{n,k}$ as in Theorem 2.1. Let $R(\tilde{N}) = r_1 + r_2$, $r_1$ and $r_2$ relatively prime. Use the Euclidean algorithm to write $r_1 = q_1r_2 + r_3$, $r_2 = q_2r_3 + r_4$, $r_3 = q_3r_4 + r_5$, $\ldots$, $r_{m-2} = q_{m-2}r_{m-1} + r_m$, $r_m = 1$. Then $r_1/r_2 = [q_1; q_2, \ldots, q_{m-2}, r_{m-1}]$, a regular continued fraction. The dyad value $R(N)$ occurs in row $n$, where

$$n = q_1 + q_2 + \cdots + q_{m-2} + r_{m-1} + 1; \tag{3.1}$$

$R(\tilde{N})$ occurs between $r_1$ and $r_2$, in columns $k$ and $(2^{n-1} + 2 - k)$, $k \geq 2$, where

$$k = 2^{q_1 + q_2 + \cdots + q_{m-2} + 1} - 2^{q_1 + q_2 + \cdots + q_{m-3} + 1} + \cdots - 2^{q_1 + q_2 + 1} + 2^{q_1 + 1}, \text{ } m \text{ odd,} \tag{3.2a}$$

or $k = 2^{q_1 + q_2 + \cdots + q_{m-2} + 1} - 2^{q_1 + q_2 + \cdots + q_{m-3} + 1} + \cdots - 2^{q_1 + 1} + 2, m \text{ even,} \tag{3.2b}$

**Proof:** Equation (3.1) is (1.2), adjusted by adding 2, since our row numbers are one more than Stern’s line numbers, and we are one row farther out. Equation (3.2a) is (1.3) when $m$ is odd, taken one row farther out; $k$ is twice the column number of $r_1$ in the $(n-1)^{st}$ row. If $r_1$ is a dyad and thus has an even column number, let $r_2 = b_{n-1,2w+1}$. If $r_1$ is the left neighbor of $r_2$, then $r_1 = b_{n-1,2w}$ and $b_{n,k} = r_1 + r_2 = b_{n,2(2w)}$; $k$ is twice the column number of $r_1$ as (3.2a). If $r_1$ is the right neighbor of $r_2$, then $r_1 = b_{n-1,2w+2}$, and $b_{n,k} = r_2 + r_1 = b_{n-1,2w+1} + b_{n-1,2w+2} = b_{2(2w+1)} = b_{n,4w+2}$, so that $k$ is 2 more than twice the column number of $r_1$ as in (3.2b).

**Lemma 3.1:** Let $b_{n,k}$ be the $k^{th}$ term of the $n^{th}$ row of the array of Theorem 2.1. The term directly below $b_{n,k}$ in the $(n+p)^{th}$ row is $b_{n,k} = b_{n+p,2p(k-1)+1}$. In particular

$$b_{1,1} = b_{1+(n-1),2^{n-1}(1-1)+1} = b_{n,1} = 1, n \geq 1;$$

$$b_{1,2} = b_{1+(n-1),2^{n-1}(2-1)+1} = b_{n,2n-1+1} = 1, n \geq 1;$$

$$b_{p,2} = b_{p+(n-p),2^{n-p}(2-1)+1} = b_{n,2n-p+1} = p, n \geq p, p = 1, 2, \ldots . \tag{3.3}$$

**Proof:** Lemma 3.1 restates Theorem 1 from [9]: If $N$ has rank $R_n$ in the $n^{th}$ line, it appears directly below in the $(n+k)^{th}$ line with rank $R_{n+k} = 2^k(R_n - 1) + 1$. □

Define a zigzag path through the array of Theorem 2.1 as movement down and right alternating with movement down and left. Define $ZR(y)$ as a movement down $y$ rows and right 1 term; $ZL(x)$, down $x$ rows and left 1 term. From Lemma 3.1,

$$ZR(y) : b_{w,t} \rightarrow b_{w+y,2^{y(t-1)+1}+1} = b_{w+y,2^{y(t-1)+2}},$$

$$ZL(x) : b_{w,t} \rightarrow b_{w+x,2^{x(t-1)+1}+1} = b_{w+x,2^{x(t-1)}}. \tag{3.4}$$
Lemma 3.2: Let $R(N) = b_{n,k} = r_1 + r_2$, $r_1/r_2 = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}]$, $r_{m-1} \geq 2$, where $r_1$ is a dyad, $r_1 > r_2$. If $b_{n,k}$ is on the left side of the table, the zigzag path from $b_{1,1}$ to $b_{n,k}$ is $ZR(r_{m-1} - 1)ZL(q_{m-2}) \cdots ZR(q_2)ZL(q_1)ZR(1)$, where $r_1$ is on the left of $r_2$, $m$ is odd, and $k$ is given by (3.2a); or, $ZR(r_{m-1} - 1)ZL(q_{m-2}) \cdots ZR(q_1)ZL(1)$, where $r_1$ is on the right of $r_2$, $m$ is even, and $k$ is given by (3.2b).

If $b_{n,k}$ is on the right side of the table, the zigzag path from $b_{1,2}$ to $b_{n,k}$ is $ZL(r_{m-1} - 1)ZR(q_{m-2}) \cdots ZL(q_2)ZR(q_1)ZL(1)$, for $r_1$ on the right of $r_2$, $m$ odd; or, $ZL(r_{m-1} - 1)ZR(q_{m-2}) \cdots ZR(q_1)ZL(1)$, $r_1$ on the left of $r_2$, $m$ even. $R(N) = b_{n,2n-1+2-k}$ for $k$ as in (3.2a) or (3.2b) as $m$ is odd or even.

Proof: On the left side of the table, the path from $b_{1,1}$ begins $ZR(r_{m-1} - 1)$ to $b_{1,1}$ followed by $ZL(q_{m-2})$. If $r_1$ is on the left of $r_2$, the path from $b_{1,1}$ will end with a move $ZR(1)$ to $R(N)$, preceded by $ZL(a_1)$ to $r_1$; $m$ is odd. If $r_1$ is on the right of $r_2$, the path from $b_{1,1}$ to $b_{n,k}$ ends $ZL(q_2)ZL(q_1)ZL(1)$, so that $m$ is even. Suppose $r_1/r_2 = [a_1; a_2, a_3, r_{m-1}]$. The zigzag path from $b_{1,1}$ to $b_{n,k}$ is $ZR(r_{m-1} - 1)ZL(a_3)ZR(a_2)ZL(a_1)ZR(1)$:

$$b_{1,1} \rightarrow b_{r_{m-1},2} \rightarrow b_{a_3+r_{m-1},2a_3(2-1)+0} \rightarrow b_{a_3+a_3+r_{m-1},2a_3(2a_3-2a_3+2-1)+0} \rightarrow b_{a_3+a_2+a_3+r_{m-1}+1,2(2a_1+a_2+a_3-a_1-a_3+2a_1+2a_1+2a_1-1)+2} = b_{n,2a_1+a_2+a_3+1-2a_1+a_2+1,2a_1+1};$$

$k$ is given by (3.2a), $a_1 = q_i$, $i = 1, 2, 3$. This pattern continues for $m$ odd. Suppose $r_1/r_2 = [a_1; a_2, r_{m-1}]$. The zigzag path from $b_{1,1}$ to $b_{n,k}$ is $ZR(r_{m-1} - 1)ZL(a_2)ZR(a_1)ZL(1)$:

$$b_{1,1} \rightarrow b_{r_{m-1},2} \rightarrow b_{a_2+r_{m-1},2a_2(2-1)+0} \rightarrow b_{a_1+a_2+r_{m-1},2a_1(2a_2-1)+2} \rightarrow b_{a_1+a_2+r_{m-1}+1,2(2a_1+a_2-2a_1+2-1)+0} = b_{n,2a_1+a_2+1-2a_1+1,2a_1+2};$$

$k$ is given by (3.2b), $a_1 = q_i$, $i = 1, 2$. The pattern continues for $m$ even.

The situation on the right side of the table is similar. The path from $b_{1,2}$ to $b_{n,k}$ on the right side is the mirror image of the path from $b_{1,1}$ to $b_{n,k}$ on the left.

Lemma 3.3: Let $R(N) = b_{n,k} = r_1 + r_2$, $r_1/r_2 = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}]$, $r_{m-1} \geq 2$. If $r_1$ is the left neighbor or $r_2$ in the $(n-1)^{st}$ row, and $m$ is odd (even), the ordered sequence, $r_1$, $R(N)$, $r_2$, appears in the $n^{th}$ row on the left (right) side of the table.

Theorem 3.2 generalizes the zigzag paths of Lemma 3.2 to $\mathcal{RLRE}$... patterns, where $R(q)$ means to write the next $(q)$ even-subscripted Fibonacci numbers; $L(q)$, omit the next $(q)$ even-subscripts. Note that $r_1$ and $r_2$ are not ordered.

Theorem 3.2: Let the dyad $R(N) = r_1 + r_2$, $r_1$ and $r_2$ relatively prime, appear in the $n^{th}$ row as in Theorem 3.1. If $R(N)$ is between $r_1$ and $r_2$ on the left side of the table, $\check{Z}eck\, \check{N}$ is given from $r_1/r_2 = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}]$, $r_{m-1} \geq 2$, by the $\mathcal{RLRE}$... pattern

$$\mathcal{R}(1)\mathcal{L}(r_{m-1} - 1)\mathcal{R}(q_{m-2})\mathcal{L}(q_{m-3}) \cdots \mathcal{R}(q_1)\mathcal{L}(1), m \text{ odd};$$

$$\cdots \mathcal{R}(q_4)\mathcal{L}(q_1)\mathcal{L}(1), m \text{ even}. \text{ The first Fibonacci number written is } F_{2n}.\quad (3.5)$$

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Proof: Let $2^q$ correspond to $F_{2(p+1)}$ as in Lemma 2.1; $R(\tilde{N}) = b_{n,k}$ is the term appearing $(k - 2)$ entries to the right of $b_{n,2} = R(F_{2n})$ where $F_{2n} = \tilde{N}(2^{n-1})$. From (3.1) with (3.2a) or (3.2b), the highest power of 2 in $k$ has exponent $(q_1 + q_2 + \cdots + q_{m-2}) = (n - r_{m-1} - 1)$. From (3.2a),

$$k - 2 = (2^{q_1 + q_2 + \cdots + q_{m-2} + 1} - 2^{q_1 + q_2 + \cdots + q_{m-3} + 1} + \cdots + (2^{q_1 + q_2 + q_3 + 1} - 2^{q_1 + q_2 + 1} + (2^{n+1} - 2)$$

$$= 2^{q_1 + q_2 + \cdots + q_m - 1}(2^{q_m - 1}) + \cdots + 2^{q_1 + q_2 + 1}(2^{q_3 - 1}) + 2(2^{q_1 - 1})$$

$$= 2^{q_1 + q_2 + \cdots + q_{m-3} + 1}(2^{q_{m-2} - 1} + \cdots + 2 + 1) + \cdots + 2^{q_1 + q_2 + 1}(2^{q_3 - 1} + \cdots + 2 + 1)$$

$$+ 2(2^{q_1 - 1} + \cdots + 1)$$

which contains $q_{m-2}$ consecutive powers of 2 beginning with $2^{q_1 + q_2 + \cdots + q_{m-2}}$, followed by $q_{m-3}$ consecutive missing powers of 2, followed by $q_{m-4}$ consecutive powers of 2, ..., ending with $q_1$ consecutive powers of 2, with the one final term $2^0$ missing. (Recall that $k$ is even, since $R(\tilde{N})$ is a dyad.) In the sum $(2^{n-1} + (k - 2))$, the leading exponent in each block of consecutive powers of 2 results from successively subtracting $r_{m-1}, q_{m-2}, q_{m-3}$ from $(n - 1)$. If $m$ is even, $(k - 2)$ as calculated from (3.2b) ends with $\cdots + (2^{q_1 + 2} - 2$, or $(q_1 + 1)$ missing powers of 2; note that $2^0$ is always missing. The pattern of (3.5) follows from Theorems 2.2 and 3.1, and Lemma 3.2. □

Corollary 3.2.1: The zigzag path in which all quotients are 1 leads to $b_{n,k} = R(\tilde{N}) = F_{n+1}$, for $\tilde{N} = F_{2n} + F_{2n-4} + F_{2n-8} + \cdots$, with smallest term $F_0$ or $F_4$, as $n$ is odd or even.

Proof: Rewrite $[1; 1, 1, \ldots, 1, 1, 1, 2]$ as $[1; 1, 1, \ldots, 1, 2]$ and use Theorem 3.2. On the right side, $\tilde{N} = F_{2n} + F_{2n-2} + F_{2n-6} + \cdots$, which results from (3.5) if $r_{m-1} = 1$. □

Corollary 3.2.2: If $R(\tilde{N})$ from Theorem 3.2 is between $r_1$ and $r_2$ on the right side of the table, Zeck $\tilde{N}$ is written from the $\mathcal{R}\mathcal{R}\mathcal{L}\mathcal{L}\mathcal{L}$... pattern, $\mathcal{R}(1)\mathcal{L}(r_{m-1} - 1)\mathcal{L}(q_{m-2})\mathcal{R}(q_{m-3})\ldots\mathcal{R}(q_1)\mathcal{L}(1)$, $m$ even; or, ending $\ldots\mathcal{L}(q_1)\mathcal{L}(1)$, $m$ odd; $r_{m-1} \geq 2$.

Proof: The zigzag path from $b_{1,2}$ to $b_{n,k}$ on the right side is the mirror image of that from $b_{1,1}$ to $b_{n,k}$ on the left side. Recall that $b_{n,k} = b_{n,2n-1+2-k}$ by symmetry. □

To illustrate, compute $\tilde{N}$ from $R(\tilde{N}) = 27 = 19 + 8$. $19/8 = [2; 2, 1, 2]$; $n = (2 + 2 + 1 + 2) + 1 = 8$, $m = 3$. We are on the left side, and Zeck $\tilde{N}$ begins $F_{18}b_{8,2} = 8$. Interpret the pattern $\mathcal{R}(1)\mathcal{L}(2)\mathcal{L}(1)\mathcal{L}(2)\mathcal{R}(2)\mathcal{L}(1)$ as use 16; omit 14; use 12; omit 10 and 8; use 6 and 4; omit 2. Thus, Zeck $\tilde{N} = F_{16} + F_{12} + F_6 + F_4 = 1142$; $R(1142) = 27$. The sequence 19, 27, 8, occurs with 27 = $b_{8,40} = R(N(j_{n,k}))$ for $j_{n,k} = 2^{8-1} + (2^2 + 2^2 + 2^1) - 2$, verifying $\tilde{N} = F_{2(7+1)} + F_{2(5+1)} + F_{2(3+1)} + F_{2(1+1)}$. On the right side, Corollary 3.2.2 gives the associated solution $\tilde{N}'$ from $\mathcal{R}(1)\mathcal{R}(2)\mathcal{L}(1)\mathcal{R}(2)\mathcal{L}(2)\mathcal{L}(1)$ as $\tilde{N}' = F_{16} + F_{14} + F_{10} + F_8 = 1440$, where $R(1440) = 27 = b_{8,90}, 8 = b_{8,89}$ and 19 = $b_{8,91}$.

The symmetries of the array for $R(\tilde{N})$ let us find other even-Zeck integers $\tilde{M}$ such that $R(\tilde{M}) = R(\tilde{N})$, with $R(\tilde{M})$ and $R(\tilde{N})$ both appearing in the $n^{th}$ row. Theorem 3.3 gives a special solution for $\tilde{M}$.
Theorem 3.3: Let $R(\bar{N}) = r_1 + r_2, r_1/r_2 = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}]$, as in Theorem 3.2; $q_1 \geq 1, r_{m-1} \geq 2$. Let Zeck $M$ be written from the $\mathcal{R}\mathcal{L}\mathcal{E}$... pattern of (3.5), adjusted by taking the quotients of $r_1/r_2$ in ascending order: $\mathcal{R}(1)\mathcal{L}(q_1)\mathcal{R}(q_2)\mathcal{L}(q_3)\ldots\mathcal{R}(r_{m-1} - 1)\mathcal{L}(1), m$ odd; $\ldots\mathcal{R}(q_{m-2})\mathcal{L}(r_{m-1} - 1)\mathcal{L}(1), m$ even. Then $R(M) = R(\bar{N})$, both appearing in row $n$.

Proof: A reversal identity for continued fractions appears as Theorem 1 in [1]: if $[a_0, a_1, \ldots, a_{n-1}, a_n] = p_n/q_n$, then $[a_n, a_{n-1}, \ldots, a_1, a_0] = p_n/p_{n-1}$. Here, $R(M) = p_n = R(\bar{N})$. □

Theorem 3.3 applied to the preceding example gives $\mathcal{R}(1)\mathcal{L}(2)\mathcal{R}(2)\mathcal{L}(1)\mathcal{R}(2 - 1)\mathcal{L}(1)$ or $M = F_{16} + F_{10} + F_8 + F_4 = 1066$; $R(1066) = 27$, but $\bar{N} = 1142$.

The Calkin and Wilf [6] sequence (1.4) is the same as our sequence (2.1); that is, $b(j) = R(\bar{N}(j))$, where $b(j)/b(j + 1)$ is the $j^{th}$ rational number, $j = 0, 1, 2, \ldots$. Thus, the results of the present paper allow us to write the $j^{th}$ rational number. Given $j$, by Lemma 2.1, we can write Zeck $\bar{N}(j)$, the Zeckendorf representation of the $j^{th}$ even-Zeck integer; there are several ways [4] to compute $R(\bar{N}(j))$ and $R(\bar{N}(j + 1))$. Given any rational number $a/b$, Theorem 3.2 can be adapted to find $\bar{N}(j)$ such that $a/b = R(\bar{N}(j))/R(\bar{N}(j + 1))$. For example, to answer at which position the rational number $13/8$ appears, place 13 between 5 and 8 in the $n^{th}$ row, $5, 13, 8; r_1/r_2 = 5/8 = [0; 1, 1, 1, 2], n = 6, m = 6$. Since $R(\bar{N}(j)) = 13$ is on the right side of the table, Corollary 3.2.2 gives $\bar{N}(j) = F_{12} + F_{10} + F_8 = 207$, and $\bar{N}(j + 1) = 208$, where $R(207) = 13, R(208) = 8$. From Zeck $\bar{N}(j), j = 2^{12/2 - 1} + 2^{10/2 - 1} + 2^{8/2 - 1} = 2^5 + 2^4 + 2^2 = 52$; thus, $13/8$ is the $52^{nd}$ rational number. Another example: to find $5/12$, use the sequence $5, 12, 7; 5/7 = [0; 1, 2, 2], n = 6, m = 3$. We are on the left side; Theorem 3.2 gives $\bar{N}(j + 1) = F_{12} + F_8 + F_4 + F_2 = 169, R(169) = 5; j = 2^{12/2 - 1} + 2^{8/2 - 1} + 2^{4/2 - 1} + 2^{2/2 - 1} = 43$. Thus, $5/12$ is the $43^{rd}$ rational number. We note that $R(\bar{N}(j))$ is another function $f(j)$ such that $f(j)/f(j + 1)$ takes every rational value exactly once, answering a question posed in [6].

![Figure 1. The Calkin-Wilk version of the tree of fractions](image-url)
Further, we can write the address of the rational number \( \frac{r_1}{r_2} \) appearing in Calkin and Wilf's tree of fractions, which is a variant of the Stern-Brocot tree [7]. The tree of fractions (Figure 1) has 1/1 at the top of the tree. Each vertex \( \frac{r_1}{r_2} \) has two children; its left child is \( \frac{r_1}{r_1 + r_2} \), and its right child is \( \frac{r_1 + r_2}{r_2} \); each fraction is \( R(\tilde{N}(j))/R(\tilde{N}(j+1)) \) for some \( j \). In the \( n \)th row of the tree, the numerators are the first 2\( n-1 \) terms of our \( n \)th row. Let \( \frac{r_1}{r_2} = [q_1; q_2, q_3, \ldots, q_{m-2}, r_{m-1}], r_m = 1, q_1 \geq 0, r_{m-1} \geq 2; \) if \( m \) is odd (even), \( r_1/r_2 \) appears on the left (right) side of the tree, and \( r_1 \) is on the left (right) of \( r_2 \) in the table. Starting from 1/1, if \( m \) is odd, the vertex \( \frac{r_1}{r_2} \) has the address \( L^{r_{m-1}}_m R_{m-2} ^{r_{m-2}} \ldots L^{r_0}_0 R^{r_0}_0 \); if \( m \) is even, \( R^{r_{m-1} - 1} L^{r_{m-2}}_m \ldots R^{r_0} L^{r_0}_0 \); where \( L^q \) means to move \( q \) vertices left; \( R^q \), move \( q \) vertices right; \( L^0 \) and \( R^0 \) are not written. If \( r_1 \) is the left neighbor of \( r_2 \) in the table and \( R(\tilde{N}(j)) = r_1 + r_2 \), then \( R(\tilde{N}(j))/R(\tilde{N}(j+1)) \) is the right child of \( r_1/r_2 \); if instead \( R(\tilde{N}(j+1)) = r_1 + r_2 \), then \( R(\tilde{N}(j))/R(\tilde{N}(j+1)) \) is the left child of \( r_1/r_2 \).

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REFERENCES

Stern's Diatomic Array Applied to Fibonacci Representations


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Tribute

JoAnn Vine

JoAnn Vine, *Fibonacci Quarterly* typist for 25 years, is retiring. She never missed a deadline and hates to give it up, but it is time to retire.

JoAnn sang with the San Francisco Opera before she married Richard Vine (*FQ* Subscription Manager for 17 years). She started her statistical typing business in 1964, typing theses for students at Stanford and San Jose State.

Thank you, JoAnn, for your years of dedicated service to the Fibonacci Association!