COMPUTATIONAL FORMULAS FOR CONVOLUTED GENERALIZED FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

In the notation of Horadam [2], let $W_n = W_n(a, b; p, q)$, where

$$W_n = pW_{n-1} - qW_{n-2} \quad (n \geq 2) \tag{1}$$

$$W_0 = a, \quad W_1 = b.$$

If α and β, assumed distinct, are the roots of

$$\lambda^2 - p\lambda + q = 0,$$

we have the Binet form

$$W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta}, \tag{2}$$

in which $A = b - a\beta$ and $B = b - a\alpha$.

The n^{th} terms of the well-known Fibonacci and Lucas numbers can be denoted by $F_n = W_n(0, 1; 1, -1)$ and $L_n = W_n(2, 1; 1, -1)$, respectively.

We also denote

$$U_n = W_n(0, 1; p, q) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \; V_n = W_n(2, p; p, q) = \alpha^n + \beta^n. \tag{3}$$

By simple computing, we have

$$\sum_{n=0}^{\infty} W_{mn} x^n = \frac{a + (bU_m - aU_{m+1})x}{1 - V_m x + q^m x^2}. \tag{4}$$

Let $W = \{W_n\}$ be defined as above, with $W_0 = 0$. For any positive integer $k \geq 2$, W. Zhang [3] obtained the following summation:

$$\sum_{a_1 + a_2 + \cdots + a_k = n} W_{a_1} W_{a_2} \ldots W_{a_k} = \frac{b^{k-1}}{(p^2 - 4q)^{k-1}(k-1)!}[g_{k-1}(n)W_{n-k+1} + h_{k-1}(n)W_{n-k}],$$

where the summation is taken over all n-tuples with positive integer coordinates $(a_1, a_2, \dots, a_k)$ such that $a_1+a_2+\cdots+a_k = n$. Moreover, $g_{k-1}(x)$ and $h_{k-1}(x)$ are two effectively computable polynomials of degree $k-1$ with coefficients only depending on p, q and k.

Recently, Z. Zhang and X. Wang [4] gave explicit expressions for $g_{k-1}(x)$ and $h_{k-1}(x)$. P. He and Z. Zhang [1] discussed generalized Lucas numbers. The purpose of this paper is to generalize the above results, i.e., to evaluate the following summation:

$$\sum_{a_1+a_2+\cdots+a_k=n} W_{ma_1} W_{ma_2} \dots W_{ma_k}.$$

2. THE CONVOLUTED FORMULA OF GENERALIZED FIBONACCI NUMBERS

Throughout this section, with $W_0 = 0$, if we let

$$G_k(x) = \left(\frac{bU_m}{1 - V_m x + q^m x^2}\right)^k = \sum_{n=0}^{\infty} W_{mn}^{(k)} x^{n-1}, \tag{5}$$

then we have

$$\sum_{a_1+a_2+\cdots+a_k=n} W_{ma_1} W_{ma_2} \dots W_{ma_k} = W_{m(n-k+1)}^{(k)}. \tag{6}$$

Theorem 2.1:

$$W_{mn}^{(k+1)} = \frac{bU_m}{k(V_m^2 - 4q^m)} \left\{ nV_M W_{m(n+1)}^{(k)} - 2q^m(n+2k-1) W_{mn}^{(k)} \right\}. \tag{7}$$

Proof: Noting that

$$\frac{d}{dx}(G_k(x)(V_m - 2q^m x)^k) = G_k'(x)(V_m - 2q^m x)^k + G_k(x) k (V_m - 2q^m x)^{k-1}(-2q^m)$$

and

$$\frac{d}{dx}(G_k(x)(V_m - 2q^m x)^k) = \frac{d}{dx}\left(\frac{bU_m(V_m - 2q^m x}{1 - V_m x + q^m x^2}\right)^k$$

$$= k\left(\frac{bU_m(V_m - 2q^m x)}{1 - V_m x + q^m x^2}\right)^{k-1} bU_m \frac{2q^m(1 - V_m x + q^m x^2) + V_m^2 - 4q^m}{(1 - V_m x + q^m x^2)^2},$$

we have

$$G'_k(x)bU_m(V_m - 2q^m x) - 2bkU_m q^m G_k(x) = 2bkU_m q^m G_k(x) + k(V_m^2 - 4q^m)G_{k+1}(x).$$

Comparing the coefficients of both sides of the equation, our theorem holds.

We denote by $\sigma_i(n,k)$ the summation of all products of choosing i elements from $n+k-i+1,\ n+k-i+2,\dots,n+2k-1$ but not containing any two consecutive elements, i.e.,

$$\sigma_i(n,k) = \sum \prod_{t=1}^{i} (n+k-i+j_t) \tag{8}$$

where the summation is taken over all i-tuples with positive integer coordinates $(j_1, j_2, \dots, j_i)$ such that $1 \le j_1 < j_2 < \dots < j_i \le k+i-1$ and $|j_r - j_s| \ge 2$ for $1 \le r \ne s \le i$.

We note that $\sigma_0(n,k) = 1$. It is easy to prove that

$$(n+2k-1)\sigma_{k-1}(n,k-1) = \sigma_k(n,k), \tag{9}$$

and

$$(n+2k-1)\sigma_{i-1}(n,k-1) + \sigma_i(n+1,k-1) = \sigma_i(n,k), \tag{10}$$

which yields

Theorem 2.2:

$$W_{mn}^{(k+1)} = \frac{(bU_m)^k}{k!(V_m^2 - 4q^m)^k} \sum_{i=0}^{k} (-2q^m)^i V_m^{k-i} \langle n \rangle_{k-i} \sigma_i(n,k) W_{m(n+k-i)}, \tag{11}$$

where $\langle n \rangle_k = n(n+1)(n+2)\dots(n+k-1)$.

Proof: We prove the theorem by induction on k. When $k = 0, 1$, the theorem is true by applying Theorem 2.1. Assume the theorem is true for a positive integer $k-1$. Then

$$W_{mn}^{(k+1)} = \frac{bU_m}{k(V_m^2 - 4q^m)}\left\{nV_m W_{m(n+1)}^{(k)} - 2q^m(n+2k-1)W_{mn}^{(k)}\right\}$$

$$= \frac{bU_m}{k(V_m^2 - 4q^m)}\{nV_m \frac{(bU_m)^{k-1}}{(k-1)!(V_m^2 - 4q^m)^{k-1}} \sum_{i=0}^{k-1}(-2q^m)^i V_m^{k-i-1}\langle n+1\rangle_{k-i-1}\times$$

$$\sigma_i(n+1, k-1)W_{m(n+k-i)} + (-2q^m)(n-1+2k)\frac{(bU_m)^{k-1}}{(k-1)!(V_m^2 - 4q^m)^{k-1}}\times$$

$$\sum_{i=0}^{k-1}(-2q^m)^i V_m^{k-i-1}\langle n\rangle_{k-i-1}\sigma_i(n, k-1)W_{m(n+k-i-1)}\}$$

$$\frac{(bU_m)^k}{(k!(V_m^2 - 4q^m)^k}\{V_m^k n\langle n+1\rangle_{k-1}\sigma_0(n+1, k-1)W_{m(n+k)}$$

$$+\sum_{i=1}^{k-1}(-2q^m)V_m^{k-i} n\langle n+1\rangle_{k-i}\sigma_i(n+1, k-1)W_{m(n+k-i)}$$

$$+\sum_{i=1}^{k}(-2q^m)^i V_m^{k-i}(n+2k-1)\langle n\rangle_{k-i}\sigma_{i-1}(n, k-1)W_{m(n+k-i)}\}$$

$$= \frac{(bU_m)^k}{k!(V_m^2 - 4q^m)^k}\{V_m^k\langle n\rangle_k\sigma_0(n, k)W_{m(n+k)}$$

$$+\sum_{i=1}^{k-1}(-2q^m)^i V_m^{k-i}\langle n\rangle_{k-i}W_{m(n+k-i)}\left[\sigma_i(n+1, k-1) + (n+2k-1)\sigma_{i-1}(n, k-1)\right]$$

$$+(-2q^m)^k(n+2k-1)\sigma_{k-1}(n, k-1)W_{mn}\}$$

(Apply (9), (10))

$$= \frac{(bU_m)^k}{k!(V_m^2-4q^m)^k}\{V_m^k\langle n\rangle_k\sigma_0(n,k)W_{m(n+k)}$$

$$+\sum_{i=1}^{k-1}(-2q^m)^iV_m^{k-i}\langle n\rangle_{k-i}\sigma_i(n,k)W_{m(n+k-i)}+(-2q^m)^k\sigma_k(n,k)W_{mn}\}$$

$$= \frac{(bU_m)^k}{k!(V_m^2-4q^m)^k}\sum_{i=0}^{k}(-2q^m)^iV_m^{k-i}\langle n\rangle_{k-i}\sigma_i(n,k)W_{m(n+k-i)}.$$

Hence the theorem is also true for k. This completes the proof.

Theorem 2.3:

$$\sum_{a_1+a_2+\cdots+a_k=n} W_{ma_1}W_{ma_2}\dots W_{ma_k}$$

$$= \frac{(bU_m)^{k-1}}{(k-1)!(V_m^2-4q^m)^{k-1}}\sum_{i=0}^{k-1}(-2q^m)^iV_m^{k-1-i}\langle n-k+1\rangle_{k-1-i}\sigma_i(n-k+1,k-1)W_{m(n-i)}. \quad (12)$$

Proof: Use (6) and Theorem 2.2.

Lemma 2.4:

$$U_mW_{m(k+n)} = U_{mn}W_{m(k+1)} - q^mU_{m(n-1)}W_{mk}. \quad (13)$$

Proof: Use (2), (3).

Let

$$g_{k-1}^{(m)}(n) = \sum_{i=0}^{k-1}(-2q^m)^iV_m^{k-1-i}\langle n-k+1\rangle_{k-1-i}\sigma_i(n-k+1,k-1)U_{m(k-i)} \quad (14)$$

and

$$h_{k-1}^{(m)}(n) = -qM\sum_{i=0}^{k-1}(-2q^m)^iV_m^{k-1-i}\langle n-k+1\rangle_{k-1-i}\sigma_i(n-k+1,k)U_{m(k-1-i)}. \quad (15)$$

Then we have the following theorem.

Theorem 2.5:

$$\sum_{a_1+a_2+\cdots+a_k=n} W_{ma_1}W_{ma_2}\dots W_{ma_k}$$

$$= \frac{(b^{k-1}U_m^{k-2}}{(k-1)!(V_m^2-4q^m)^{k-1}}\left\{g_{k-1}^{(m)}(n)W_{m(n-k+1)}+h_{k-1}^{(m)}(n)W_{m(n-k)}\right\}. \tag{16}$$

Proof: Use Theorem 2.3 and Lemma 2.4.

Corollary 2.6:

$$\sum_{a+b=n} W_{ma}W_{mb}$$

$$= \frac{b}{V_m^2-4q^m}\{[(n-1)V_mU_{2m}-2q_mnU_m]W_{m(n-1)}-q^m(n-1)V_mU_mW_{m(n-2)}\},$$

$$\sum_{a+b+c=n} W_{ma}W_{mb}W_{mc}$$

$$\frac{b^2U_m}{2\left(V_m^2-4q^m\right)^2}\left\{(n-2)(n-1)V_m^2U_{3m}-2q_m(n-2)(2n+1)V_mU_{2m}+4q^{2m}(n-1)\right.$$

$$\left.(n+1)U_m]W_{m(n-2)}-q^m[(n-2)(n-1)V_m^2U_{2m}-2q^m(n-2)(2n+1)V_mU_m]W_{m(n-3)}\right\}.$$

Proof: Take $k=2,3$ in Theorem 2.5.

From (16), we have

Corollary 2.7:

$$b^{k-1}U_m^{k-2}\left\{g_{k-1}^{(m)}(n)W_{m(n-k+1)}+h_{k-1}^{(m)}(n)W_{m(n-k)}\right\}\equiv 0 \pmod{(k-1)!(V_m^2-4q^m)^{k-1}}. \tag{17}$$

3. THE CONVOLUTED FORMULA OF GENERALIZED LUCAS NUMBERS

Let $\Delta = 2U_{m+1}-pU_m$. Taking $a=2,\ b=p$ and using (4) we have

$$\sum_{n=0}^{\infty} V_{mn}x^n = \frac{2-\Delta x}{1-V_mx+q^mx^2}.$$

Let

$$H_k(x) = \sum_{n=0}^{\infty} V_{mn}^{(k)}x^n = \left(\frac{2-\Delta x}{1-V_mx+q^mx^2}\right)^k.$$

Obviously, $V_{mn}^{(1)} = V_{mn}$. From these, we have

$$\sum_{a_1+a_2+\cdots+a_k=n} V_{ma_1} V_{ma_2} \ldots V_{ma_k} = V_{mn}^{(k)}. \tag{18}$$

Theorem 3.1:

$$k(\Delta V_m - 4q^m)V_{mn}^{(k+1)} = 4(n+2)V_{m(n+2)}^{(k)} - 2(2n+k+2)\Delta V_{m(n+1)}^{(k)} + (n+k)\Delta^2 V_{mn}^{(k)}. \tag{19}$$

Proof: Noting that

$$\frac{d}{dx}(H_k(x) = \frac{d}{dx}\left(\frac{2-\Delta x}{1-V_m x + q^m x^2}\right)^k$$

$$= k\left(\frac{2-\Delta x}{1-V_m x + q^m x^2}\right)^{k-1} \frac{-\Delta + 2V_m - 4q^m x + q^m \Delta x^2}{(1-V_m x + q^m x^2)^2}$$

$$= k\left(\frac{2-\Delta x}{1-V_m x + q^m x^2}\right)^{k-1} \frac{\Delta[1-V_m x + q^m x^2] + [\Delta V_m - 4q^m}{(1-V_m x + q^m x^2)^2}$$

$$= k\frac{\Delta}{2-\Delta x}\left(\frac{2-\Delta x}{1-V_m x + q^m x^2}\right)^k + k\frac{\Delta V_m - 4q^m}{(2-\Delta x)^2}x\left(\frac{2-\Delta x}{(1-V_m x + q^m x^2)}\right)^{k+1},$$

we have

$$(2-\Delta x)^2 \frac{d}{dx}H_k(x) = k[\Delta V_m - 4q^m]xH_{k+1}(x) + k\Delta(2-\Delta x)H_k(x).$$

This implies

$$k[\Delta V_m - 4q^m]xH_{k+1}(x) = (4 - 4\Delta x + \Delta^2 x^2)\frac{d}{dx}H_k(x) - k\Delta(2-\Delta x)H_k(x).$$

Comparing the coefficients of both sides in the above equation, the theorem holds.

Theorem 3.2:

$$\sum_{a+b=n} V_{ma}V_{mb}$$

$$= \frac{1}{\Delta V_m - 4q^m}\left\{4(n+2)V_{m(n+2)} + 2(2n+3)\Delta V_{m(n+1)} + (n+1)\Delta^2 V_{mn}\right\}. \tag{20}$$

$$\sum_{a+b+c=n} V_{ma}V_{mb}V_{mc} = \frac{1}{2(\Delta V_m - 4q^m)^2}\{16(n+4)(n_+2)V_{m(n+4)}$$

$$+8((n+2)(2n+7)+(n+3)(2n+4))\Delta V_{m(n+3)} + 4((n+2)(n+3)+(2n+4)(2n+5)$$

$$+(n+2)^2)\Delta^2 V_{m(n+2)} + 2(n+2)(4n+7)\Delta^3 V_{m(n+1)} + (n+1)(n+2)\Delta^4 V_{mn}\}. \tag{21}$$

Proof: Use (18) and Theorem 3.1.

In Theorem 3.1, taking $m = 1, 2$ gives the main results of paper [1].

Corollary 3.3:

$$k(p^2-4q)V_n^{(k+1)} = 4(n+2)V_{n+2}^{(k)} - 2p(2n+k+2)V_{n+1}^{(k)} + p^2(n+k)V_n^{(k)}, \tag{22}$$

$$kp^2(p^2-4q)V_{2n}^{(k+1)} = 4(n+2)V_{2(n+2)}^{(k)} - 2(2n+k+2)(p^2-2q)V_{2(n+1)}^{(k)} + (n+k)(p^2-2q)^2 V_{2n}^{(k)}. \tag{23}$$

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