GENERALIZED FIBONACCI FUNCTIONS AND SEQUENCES OF GENERALIZED FIBONACCI FUNCTIONS

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1. INTRODUCTION

We consider a generalization of the Fibonacci sequence which is called the $k$-Fibonacci sequence for a positive integer $k \geq 2$. The $k$-Fibonacci sequence $\{g_n^{(k)}\}$ is defined as

$$g_0^{(k)} = g_1^{(k)} = \cdots = g_{k-2}^{(k)} = 0, \quad g_{k-1}^{(k)} = 1$$

and for $n \geq k \geq 2$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}.$$

We call $g_n^{(k)}$ the $n$th $k$-Fibonacci number. For example, if $k = 2$, then $\{g_n^{(2)}\}$ is the Fibonacci sequence $\{F_n\}$. If $k = 5$, then $g_0^{(5)} = g_1^{(5)} = g_2^{(5)} = g_3^{(5)} = 0$, $g_4^{(5)} = 1$, and the 5-Fibonacci sequence is

$$\left( g_0^{(5)} = 0 \right), 0, 0, 0, 1, 1, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, \ldots.$$

Let $E$ be a 1 by $(k-1)$ matrix whose entries are ones and let $I_n$ be the identity matrix of order $n$. Let $g_n^{(k)} = (g_n^{(k)}, \ldots, g_{n+k-1}^{(k)})^T$ for $n \geq 0$. For any $k \geq 2$, the fundamental recurrence relation, $n \geq k$,

$$g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}$$

can be defined by the vector recurrence relation $g_{n+1}^{(k)} = Q_k g_n^{(k)}$, where

$$Q_k = \begin{bmatrix} 0 & I_{k-1} \\ 1 & E \end{bmatrix}. \quad (1)$$
We call $Q_k$ the $k$-Fibonacci matrix. By applying (1), we have $g_{n+1}^{(k)} = Q_k^n g_1^{(k)}$. In [4], [6] and [7], we can find relationships between the $k$-Fibonacci numbers and their associated matrices.

In [2], M. Elmore introduced the Fibonacci function following as:

$$f_0(x) = \frac{e^{\lambda_1 x} - e^{\lambda_2 x}}{\sqrt{5}}, \quad f_n(x) = f_0^{(n)}(x) = \frac{\lambda_1^n e^{\lambda_1 x} - \lambda_2^n e^{\lambda_2 x}}{\sqrt{5}},$$

and hence $f_{n+1}(x) = f_n(x) + f_{n-1}(x)$, where

$$\lambda_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1 - \sqrt{5}}{2}.$$

Here, $\lambda_1, \lambda_2$ are the roots of $x^2 - x - 1 = 0$.

In this paper, we consider a function which is a generalization of the Fibonacci function and consider sequences of generalized Fibonacci functions.

## 2. GENERALIZED FIBONACCI FUNCTIONS

For positive integers $l$ and $n$ with $l \leq n$, let $Q_{l,n}$ denote the set of all strictly increasing $l$-sequences from $\{1, 2, \ldots, n\}$. For an $n \times n$ matrix $A$ and for $\alpha, \beta \in Q_{l,n}$, let $A[\alpha|\beta]$ denote the matrix lying in rows $\alpha$ and columns $\beta$ and let $A(\alpha|\beta)$ denote the matrix complementary to $A[\alpha|\beta]$ in $A$. In particular, we denote $A(\{i\}|\{j\}) = A(i|j)$.

We define a function $G(k, x)$ by

$$G(k, x) = \sum_{i=0}^{\infty} \frac{g_i^{(k)}}{i!} x^i.$$

Since

$$\lim_{n \to \infty} \frac{g_n^{(k)}(n+1)}{g_{n+1}^{(k)}} \to \infty,$$

the function $G(k, x)$ is convergent for all real number $x$.

For fixed $k \geq 2$, the power series $G(k, x)$ satisfies the differential equation

$$G^{(k)}(k, x) - G^{(k-1)}(k, x) - \cdots - G''(k, x) - G'(k, x) - G(k, x) = 0. \quad (2)$$

In [5], we can find that the characteristic equation $x^k - x^{k-1} - \cdots - x - 1 = 0$ of $Q_k$ does not have multiple roots. So, if $\lambda_1, \lambda_2, \ldots, \lambda_k$ are the roots of $x^k - x^{k-1} - \cdots - x - 1 = 0$, then
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$\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. That is, the eigenvalues of $Q_k$ are distinct. Define $V$ to be the $k$ by $k$ Vandermonde matrix by

$$V = \begin{bmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_k \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{k-2} & \lambda_2^{k-2} & \cdots & \lambda_k^{k-2} \\
\lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1}
\end{bmatrix}.$$  \hspace{1cm} (2)

Then we have the following theorem.

**Theorem 2.1:** Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the eigenvalues of the $k$-Fibonacci matrix $Q_k$. Then, the initial-value problem $\sum_{i=0}^{k-1} G^{(i)}(k, x) = G^{(k)}(k, x)$, where $G^{(i)}(k, 0) = 0$ for $i = 0, 1, \ldots, k - 2$, and $G^{(k-1)}(k, 0) = 1$ has the unique solution $G(k, x) = \sum_{i=1}^{k} c_i e^{\lambda_i x}$, where

$$c_i = (-1)^{k+i} \frac{\det V(k | i)}{\det V}, \quad i = 1, 2, \ldots, k.$$ \hspace{1cm} (3)

**Proof:** Since the characteristic equation of $Q_k$ is $x^k - x^{k-1} - \cdots - x - 1 = 0$, it is clear that $c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_k e^{\lambda_k x}$ is a solution of (2).

Now, we will prove that $c_i = \frac{\det V}{\det V} (-1)^{k+i} \det V(k | i), i = 1, 2, \ldots, k$. Since $G(k, x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \cdots + c_k e^{\lambda_k x}$ and for $x = 0$, $G^{(0)}(k, 0) = 0$ for $i = 0, 1, \ldots, k - 2, G^{(k-1)}(k, 0) = 1$, we have

$$G(k, 0) = c_1 + c_2 + \cdots + c_k = 0$$
$$G'(k, 0) = c_1 \lambda_1 + c_2 \lambda_2 + \cdots + c_k \lambda_k = 0$$
$$\vdots$$

$$G^{(k-2)}(k, 0) = c_1 \lambda_1^{k-2} + c_2 \lambda_2^{k-2} + \cdots + c_k \lambda_k^{k-2} = 0$$
$$G^{(k-1)}(k, 0) = c_1 \lambda_1^{k-1} + c_2 \lambda_2^{k-1} + \cdots + c_k \lambda_k^{k-1} = 1.$$

Let $c = (c_1, c_2, \ldots, c_{k-1}, c_k)^T$ and $b = (0, 0, \ldots, 0, 1)^T$. Then we have $Vc = b$. Since the matrix $V$ is a Vandermonde matrix and $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct, the matrix $V$ is nonsingular. For $i = 1, 2, \ldots, k$, the matrix $V(k | i)$ is also a Vandermonde matrix and nonsingular. Therefore, by Cramer’s rule, we have $c_i = (-1)^{k+i} \frac{\det V(k | i)}{\det V}, i = 1, 2, \ldots, k$ and the proof is complete. 

We can replace the writing of (2) by the form

$$G^{(k)}(k, x) = G^{(k-1)}(k, x) + \cdots + G''(k, x) + G'(k, x) + G(k, x).$$

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This suggests that we use the notation \( G_0(k, x) = G(k, x) \) and, for \( i \geq 1 \), \( G_i(k, x) = G^{(i)}(k, x) \). Thus

\[
G_n(k, x) = G^{(n)}(k, x) = c_1 \lambda_1^n e^{\lambda_1 x} + c_2 \lambda_2^n e^{\lambda_2 x} + \cdots + c_k \lambda_k^n e^{\lambda_k x}
\]

gives us the sequence of functions \( \{G_n(k, x)\} \) with the property that

\[
G_n(k, x) = G_{n-1}(k, x) + G_{n-2}(k, x) + \cdots + G_{n-k}(k, x), \quad n \geq k,
\]

where each \( c_i \) is in (3). We shall refer to these functions as \( k \)-Fibonacci functions. If \( k = 2 \), then \( G(2, x) = f_0(x) \) is the Fibonacci function as in [2]. From (4), we have the following theorem.

**Theorem 2.2:** For the \( k \)-Fibonacci function \( G_n(k, x) \),

\[
G_0(k, 0) = 0 = g_0^{(k)}, G_1(k, 0) = 0 = g_1^{(k)}, \ldots, G_{k-2}(k, 0) = 0 = g_{k-2}^{(k)},
\]

\[
G_{k-1}(k, 0) = 1 = g_{k-1}^{(k)}, G_k(k, 0) = G_0(k, 0) + \cdots + G_{k-1}(k, 0) = 1 = g_k^{(k)},
\]

\[
g_n^{(k)} = G_n(k, 0) = c_1 \lambda_1^n + c_2 \lambda_2^n + \cdots + c_k \lambda_k^n = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)}, \quad n \geq k,
\]

where each \( c_i \) is given by (3).

Let \( G_n(k, x) = (G_n(k, x), \ldots, G_{n+k-1}(k, x))^T \). For \( k \geq 2 \), the fundamental recurrence relation (4) can be defined by the vector recurrence relation \( G_{n+1}(k, x) = Q_k G_n(k, x) \) and hence \( G_{n+1}(k, x) = Q_k^n G_1(k, x) \).

Since \( g_{k-1}^{(k)} = g_k^{(k)} = 1 \), we can replace the matrix \( Q_k \) in (1) with

\[
Q_k = \begin{bmatrix}
0 & g_0^{(k)} & 0 & \cdots & 0 \\
0 & 0 & g_1^{(k)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & g_{k-2}^{(k)} \\
g_{k-1}^{(k)} & g_k^{(k)} & \cdots & g_{k-2}^{(k)} & g_{k-1}^{(k)}
\end{bmatrix}.
\]

Then we can find the matrix \( Q_k^n = [g_{i,j}^n(n)] \) in [5] where, for \( i = 1, 2, \ldots, k \) and \( j = 1, 2, \ldots, k \),

\[
g_{i,j}^n(n) = g_{n+i-2}^{(k)} + \cdots + g_{n+i-(j-1)}^{(k)}.
\]

We know that \( g_{i,1}^n(n) = g_{n+i-2}^{(k)} \) and \( g_{k,k}^n(n) = g_{n+i-1}^{(k)} \). So, we have the following theorem.
Theorem 2.3: For nonnegative integers $n$ and $m$, $n + m \geq k$, we have

$$G_{n+m+1}(k, x) = \sum_{j=1}^{k} g_{1,j}^{\uparrow}(n) G_{m+j}(k, x).$$

In particular,

$$G_k(k, x) = \sum_{i=0}^{\infty} \frac{g_{i+k}^{(k)}}{i!} x^i.$$

Proof: Since $G_{n+1}(k, x) = Q^k_k G_1(k, x)$,

$$G_{n+m+1}(k, x) = Q_k^{n+m} G_1(k, x) = Q_k^{n} \cdot Q_k^{m} G_1(k, x) = Q_k^{n+m+1}(k, x).$$

By applying (5), we have

$$G_{n+m+1}(k, x) = g_{1,1}^{\uparrow}(n) G_{m+1}(k, x) + \cdots + g_{1,k}^{\uparrow}(n) G_{m+k}(k, x).$$

Since $\sum_{i=0}^{k-1} G_i(k, x) = G_k(k, x)$ and

$$\sum_{i=0}^{k-1} G_i(k, x) = g_k^{(k)} + g_{k+1}^{(k)} x + \frac{g_{k+2}^{(k)}}{2!} x^2 + \cdots + \frac{g_{n+k}^{(k)}}{n!} x^n + \cdots,$$

we have

$$G_k(k, x) = \sum_{i=0}^{\infty} \frac{g_{i+k}^{(k)}}{i!} x^i. \quad \square$$

Note that $Q_k^{n+m} = Q_k^{m+n}$. Then we have the following corollary.

Corollary 2.4: For nonnegative integers $n$ and $m$, $n + m \geq k$, we have

$$G_{n+m+1}(k, x) = \sum_{j=1}^{k} g_{1,j}^{\uparrow}(m) G_{n+j}(k, x).$$

We know that the characteristic polynomial of $Q_k$ is $\lambda^k - \lambda^{k-1} - \cdots - \lambda - 1$. So, we have the following lemma.
Lemma 2.5: Let \( \lambda^k - \lambda^{k-1} - \cdots - \lambda - 1 = 0 \) be the characteristic equation of \( Q_k \). Then, for any root \( \lambda \) of the characteristic equation, \( n \geq k > 0 \), we have,

\[
\lambda^n = \sum_{j=1}^{k} g_{1,j}(n)\lambda^{j-1}.
\]

**Proof:** From (5) we have, for \( j = 1, 2, \ldots, k \),

\[
g_{1,j}(n) = g_{n-1}^{k} + g_{n-2}^{k} + \cdots + g_{n-j}^{k}.
\]

It can be shown directly for \( n = k \) that

\[
\lambda^k = g_{n-1}^{(k)}\lambda^{k-1} + \left( g_{n-2}^{(k)} + g_{n-3}^{(k)} + \cdots + g_{n-k+1}^{(k)} \right)\lambda^{k-2} + \cdots + \left( g_{n-k+1}^{(k)} + g_{n-k+2}^{(k)} \right)\lambda + g_{n-k+1}^{k} - 1 + \lambda + 1
\]

We show this by induction on \( n \). Then

\[
\lambda^{n+1} = \lambda^n \cdot \lambda
\]

\[
= \left( g_{1,k}(n)\lambda^{k-1} + g_{1,k-1}(n)\lambda^{k-2} + \cdots + g_{1,2}(n)\lambda + g_{1,1}(n) \right) \lambda
\]

\[
= g_{n}^{k}\lambda + \left( g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k+1}^{(k)} \right)\lambda^{k-1}
\]

\[
+ \left( g_{n-1}^{(k)} + \cdots + g_{n-k+2}^{(k)} \right)\lambda^{k-2} + \cdots + \left( g_{n-1}^{(k)} + g_{n-2}^{(k)} \right)\lambda^2 + g_{n-1}^{k} \lambda.
\]

Since \( \lambda^k = \lambda^{k-1} + \cdots + \lambda + 1 \), we have
\[ \lambda^{n+1} = g^{(k)}_n (\lambda^{k-1} + \ldots + \lambda + 1) + \left( g^{(k)}_{n-1} + g^{(k)}_{n-2} + \ldots + g^{(k)}_{n-k+1} \right) \lambda^{k-1} + \]
\[ \left( g^{(k)}_{n-1} + g^{(k)}_{n-2} + \ldots + g^{(k)}_{n-k+2} \right) \lambda^{k-2} + \ldots + \left( g^{(k)}_{n-1} + g^{(k)}_{n-2} \right) \lambda^2 + g^{(k)}_{n-1} \lambda \]
\[ = \left( g^{(k)}_n + g^{(k)}_{n-1} + \ldots + g^{(k)}_{n-k+1} \right) \lambda^{k-1} + \left( g^{(k)}_n + \ldots + g^{(k)}_{n-k+2} \right) \lambda^{k-2} \]
\[ + \ldots + \left( g^{(k)}_n + g^{(k)}_{n-1} \right) \lambda + g^{(k)}_n \]
\[ = g^{(k)}_{n+1} \lambda^{k-1} + \left( g^{(k)}_n + g^{(k)}_{n-1} + \ldots + g^{(k)}_{n-k+2} \right) \lambda^{k-2} \]
\[ + \ldots + \left( g^{(k)}_n + g^{(k)}_{n-1} \right) \lambda + g^{(k)}_n \]
\[ = g_{1,k}^{(1)}(n + 1)\lambda^{k-1} + g_{1,k-1}^{(1)}(n + 1)\lambda^{k-2} + g_{1,k-2}^{(1)}(n + 1)\lambda^{k-3} \]
\[ + \ldots + g_{1,2}^{(1)}(n + 1)\lambda + g_{1,1}^{(1)}(n + 1) \]
\[ = \sum_{j=1}^{k} g_{1,j}^{(1)}(n + 1)\lambda^{j-1}. \]

Therefore, by induction of \( n \), the proof is completed. \( \square \)

**Theorem 2.6:** Let \( \lambda \) be a root of characteristic equation of \( Q_k \). For positive integer \( n \), we have

\[ G_n(k, \lambda) = \sum_{j=n}^{k} \alpha_{nj} \lambda^{j-1}, \]

where

\[ \alpha_{j,n} = \frac{g_{n+k}^{(k)}}{k!} + \frac{g_{n+j-1}^{(k)}}{(j-1)!} + \sum_{i=k+1}^{\infty} g_{1,j}^{(1)}(i) \frac{g_{n+i}^{(k)}}{i!}. \]

**Proof:** Since \( \lambda^k = \lambda^{k-1} + \ldots + \lambda + 1 \) and by lemma 2.5, we have
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\[ G_n(k, \lambda) = g_n^{(k)} + g_{n+1}^{(k)} \lambda + \frac{g_{n+2}^{(k)}}{2!} \lambda^2 + \ldots + \frac{g_{2n}^{(k)}}{n!} \lambda^n + \ldots \]

\[ = \left( g_n^{(k)} + \frac{g_{n+k}^{(k)}}{k!} + g_{11}^{(k)} (k+1) \frac{g_{n+k+1}^{(k)}}{(k+1)!} + \ldots + g_{11}^{(n)} (n) \frac{g_{2n}^{(k)}}{n!} + \ldots \right) + \]

\[ \left( g_{n+1}^{(k)} + \frac{g_{n+k}^{(n+k)}}{k!} + g_{12}^{(k)} (k+1) \frac{g_{n+k+1}^{(k)}}{(k+1)!} + \ldots + g_{12}^{(n)} (n) \frac{g_{2n}^{(k)}}{n!} + \ldots \right) \lambda \]

\[ + \cdots + \]

\[ \left( \frac{g_{n+k-1}^{(k)}}{(k-1)!} + \frac{g_{n+k}^{(k)}}{k!} + g_{1k}^{(k)} (k+1) \frac{g_{n+k+1}^{(k)}}{(k+1)!} + \ldots + g_{1k}^{(n)} (n) \frac{g_{2n}^{(k)}}{n!} + \ldots \right) \lambda^{k-1} \]

\[ = \alpha_1 \lambda + \alpha_2 \lambda + \ldots + \alpha_k \lambda^{k-1} \]

\[ = \sum_{j=1}^{k} \alpha_j \lambda^{j-1}, \]

where

\[ \alpha_j = \frac{g_{n+k}^{(k)}}{k!} + g_{n+j-1}^{(k)} + \sum_{i=k+1}^{\infty} g_{i,j}^{(i)} (i) \frac{g_{n+i}^{(k)}}{i!} \]

for \( j = 1, 2, \ldots, k \), the proof is completed. \( \square \)

From theorem 2.3 and theorem 2.6, we have

\[ G_n(k, x) = \sum_{i=0}^{\infty} \frac{g_{n+i}^{(k)}}{i!} x^i \]

\[ = g_{1,1}^{(k)} (n-1) G_1(k, x) + \ldots + g_{1,k}^{(k)} (n-1) G_k(k, x) \]

\[ = \sum_{j=1}^{k} \alpha_j \lambda^{j-1}, \]
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where

\[ \alpha_j = \frac{g_{n+k}^{(k)}}{k!} + \frac{g_{n+j-1}^{(k)}}{(j-1)!} + \sum_{i=k+1}^{\infty} g_{1,i}^{(k)} \frac{g_{n+i}^{(k)}}{i!} \]

for \( j = 1, 2, \ldots, k \).

3. SEQUENCES OF GENERALIZED FIBONACCI FUNCTIONS

Matrix methods are a major tool in solving certain problems stemming from linear recurrence relations. In this section, the procedure will be illustrated by means of a sequence, and an interesting example will be given.

To begin with, we introduce the concept of the resultant of given polynomials [3]. Let \( f(x) = \sum_{i=0}^{n} a_ix^{n-i} \) and \( g(x) = \sum_{i=0}^{m} b_ix^{m-i} \) be polynomials, where \( a_0 \neq 0 \) and \( b_0 \neq 0 \). The presence of a common divisor for \( f(x) \) and \( g(x) \) is equivalent to the fact that there exists polynomials \( p(x) \) and \( q(x) \) such that \( f(x)q(x) = g(x)p(x) \) where \( \text{deg } p(x) \leq n - 1 \) and \( \text{deg } q(x) \leq m - 1 \). Let \( g(x) = u_0x^{m-1} + \cdots + u_{m-1} \) and \( p(x) = v_0x^{n-1} + \cdots + v_{n-1} \). The equality \( f(x)q(x) = g(x)p(x) \) can be expressed in the form of a system of equations

\[
\begin{align*}
0 & \quad u_0v_0 \\
1 & \quad a_1u_0 + a_0u_1 = b_1v_0 + b_0v_1 \\
\vdots & \quad \quad \vdots \\
0 & \quad a_2u_0 + a_1u_1 + a_0u_2 = b_2v_0 + b_1v_1 + b_0v_2 \\
\end{align*}
\]

The polynomials \( f(x) \) and \( g(x) \) have a common root if and only if this system of equations has a nonzero solution \( (u_0, u_1, \ldots, v_0, v_1, \ldots) \). If, for example, \( m = 3 \) and \( n = 2 \), then the determinant of this system is of the form

\[
\begin{vmatrix}
0 & 0 & 0 & -b_0 & 0 \\
0 & 0 & a_0 & a_1 & a_2 & 0 \\
0 & 0 & a_0 & a_1 & a_2 & 0 \\
0 & 0 & a_0 & a_1 & -b_3 & -b_2 & 0 \\
0 & 0 & a_0 & a_1 & -b_3 & -b_2 & 0 \\
0 & 0 & a_0 & a_1 & -b_3 & -b_2 & 0 \\
\end{vmatrix} = |S(f(x), g(x))|.
\]

The matrix \( S(f(x), g(x)) \) is called the Sylvester matrix of polynomials \( f(x) \) and \( g(x) \). The determinant \( |S(f(x), g(x))| \) is called the resultant of \( f(x) \) and \( g(x) \) and is denoted by \( R(f(x), g(x)) \). It is clear that \( R(f(x), g(x)) = 0 \) if and only if the polynomials \( f(x) \) and \( g(x) \) have a common divisor, and hence, an equation \( f(x) = 0 \) has multiple roots if and only if \( R(f(x), f'(x)) = 0 \).

Now, we define a sequence. For fixed \( k, k \geq 2 \), and a complex number \( a \), a sequence of \( k \)-Fibonacci functions, \( \{G_n(k, a)\} \), is defined recursively as follows:

\[ G_0(k, a) = s_0, \quad G_1(k, a) = s_1, \quad \ldots, \quad G_{k-1}(k, a) = s_{k-1}, \quad (6) \]

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\[ G_n(k, a) = p_1 G_{n-1}(k, a) + p_2 G_{n-2}(k, a) + \cdots + p_k G_{n-k}(k, a), \quad n \geq k, \]

where \( s_0, s_1, \ldots, s_{k-1}, p_1, p_2, \ldots, p_k \) are complex numbers.

Our natural question now becomes, for \( k \geq 2 \), what is an explicit expression for \( G_n(k, a) \) is terms of \( s_0, s_1, \ldots, s_{k-1}, p_1, \ldots, p_k \) If \( s_0 = \cdots = s_{k-2} = 0, s_{k-1} = s_k = 1, p_1 = \cdots = p_k = 1 \) and \( a = 0 \), then by theorem 2.2 we have \( G_n(k, 0) = g_n \). In [8], Rosenbaum gave the explicit expression for \( k = 2 \).

In this section, we give an explicit expression for \( G_n(k, a) = p_1 G_{n-1}(k, a) + p_2 G_{n-2}(k, a) + \cdots + p_k G_{n-k}(k, a), \quad n \geq k \) in terms of initial conditions \( G_0(k, a) = s_0, G_1(k, a) = s_1, \ldots, G_{k-1}(k, a) = s_{k-1}, k \geq 2 \).

Let \( \vec{G}_n(k) = (G_n(k, a), \ldots, G_{n-k+1}(k, a))^T \) for \( k \geq 2 \). The fundamental recurrence relation \( \vec{G}_n(k) = \vec{Q}_k \vec{G}_{n-1}(k) \), where

\[ \vec{Q}_k = \begin{bmatrix} P & p_k \\ I_{k-1} & 0 \end{bmatrix} \text{ and } p = [p_1, p_2, \ldots, p_{k-1}] \]

Let \( s = (s_{k-1}, \ldots, s_0)^T \). Then, we have, for \( n \geq 0 \), \( \vec{G}_{n+k-1}(k) = \vec{Q}_k^n s \), and the characteristic equation of \( \vec{Q}_k \) is

\[ f(\lambda) = \lambda^k - p_1 \lambda^{k-1} - \cdots - p_{k-1} \lambda - p_k = 0. \]

If \( R(f(\lambda), f'(\lambda)) \neq 0 \), then the equation \( f(\lambda) = 0 \) has distinct \( k \) roots.

Theorem 3.1: Let \( f(\lambda) \) be the characteristic equation of the matrix \( \vec{Q}_k \). If \( R(f(\lambda), f'(\lambda)) \neq 0 \), then \( G_n(k, a) = p_1 G_{n-1}(k, a) + p_2 G_{n-2}(k, a) + \cdots + p_k G_{n-k}(k, a) \) has an explicit expression in terms of \( s_0, \ldots, s_{k-1} \).

Proof: If \( R(f(\lambda), f'(\lambda)) \neq 0 \), then the characteristic equation of \( \vec{Q}_k \) has \( k \) distinct roots, say \( \lambda_1, \lambda_2, \ldots, \lambda_k \). Since the matrix \( \vec{Q}_k \) is diagonalizable, there exists a matrix \( \Lambda \) such that \( \Lambda^{-1} \vec{Q}_k \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k) \). Then \( \vec{G}_{n+k-1}(k) = \Lambda \text{diag}(\lambda_1^n, \lambda_2^n, \ldots, \lambda_k^n) \Lambda^{-1} s \), and hence we have

\[ G_n(k, a) = d_1 \lambda_1^n + d_2 \lambda_2^n + \cdots + d_k \lambda_k^n = \sum_{i=1}^{k} d_i \lambda_i^n, \]

where \( d_1, d_2, \ldots, d_k \) are complex numbers independent of \( n \). We can determine the values of \( d_1, d_2, \ldots, d_k \) by Cramer’s rule. That is, by setting \( n = 0, 1, \ldots, k-1 \), we have

\[ G_0(k, a) = d_1 + d_2 + \cdots + d_k, \]
\[ G_1(k, a) = d_1 \lambda_1 + d_2 \lambda_2 + \cdots + d_k \lambda_k, \]
\[ \vdots \]
\[ G_{k-1}(k, a) = d_1 \lambda_1^{k-1} + d_2 \lambda_2^{k-1} + \cdots + d_k \lambda_k^{k-1}, \]
and hence

\[ V_d = s, \quad d = (d_1, d_2, \ldots, d_k)^T. \tag{8} \]

Therefore, we now have the desired result from (8). \( \square \)

Recall that

\[ \tilde{Q}_k = \begin{bmatrix} p & p_k \\ I_{k-1} & 0 \end{bmatrix}, \]

where \([p = p_1, p_2, \ldots, p_{k-1}]\). Then, in [1], we have the following theorem.

**Theorem 3.2** [1]: The \((i, j)\) entry \(q^{(n)}_{ij}(p_1, p_2, \ldots, p_k)\) in \(\tilde{Q}_k^n\) is given by the following formula:

\[
q^{(n)}_{ij}(p_1, p_2, \ldots, p_k) = \sum_{(m_1, \ldots, m_k)} \frac{m_j + m_{j+1} + \cdots + m_k}{m_1 + \cdots + m_k} \\
\times \left( \frac{m_1 + \cdots + m_k}{m_1, m_2, \ldots, m_k} \right) p_1^{m_1} \cdots p_k^{m_k}, \tag{9}
\]

where the summation is over nonnegative integers satisfying \(m_1 + 2m_2 + \cdots + km_k = n - i + j\), and the coefficient in (9) is defined to be 1 if \(n = i - j\).

Applying the \(G_{n+k-1}(k) = \tilde{Q}_k^n\) to the above theorem, we have

\[
G_n(k, a) = q^{(n)}_{k1}(p_1, \ldots, p_k)s_{k-1} + q^{(n)}_{k2}(p_1, \ldots, p_k)s_{k-2} + \\
\cdots + q^{(n)}_{kk}(p_1, \ldots, p_k)s_0 \\
= \sum_{j=1}^{k} q^{(n)}_{kj}(p_1, \ldots, p_k)s_{k-j}. \tag{10}
\]

From (9), we have

\[
q^{(n)}_{kj}(p_1, \ldots, p_k) = \sum_{(m_1, \ldots, m_k)} \frac{m_j + m_{j+1} + \cdots + m_k}{m_1 + \cdots + m_k} \\
\times \left( \frac{m_1 + \cdots + m_k}{m_1, m_2, \ldots, m_k} \right) p_1^{m_1} \cdots p_k^{m_k},
\]

where the summation is over nonnegative integers satisfying \(m_1 + 2m_2 + \cdots + km_k = n - k + j\), and the coefficient in (10) is defined to be 1 if \(n = k - j\).
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Hence, from theorem 3.1 and (10),

\[ G_n(k, a) = \sum_{j=1}^{k} q_{kj}^{(n)} (p_1, \ldots, p_k) s_{k-j} \]

\[ = \sum_{i=1}^{k} d_i \lambda_i^n. \]

**Example:** In (6) and (7), if we take \( a = 0, s_0 = s_1 = \cdots = s_{k-3} = 0, s_{k-2} = s_{k-1} = 1 \) and \( p_1 = \cdots = p_k = 1 \), then

\[ G_0(k, 0) = \cdots = G_{k-3}(k, 0) = 0, \ G_{k-2}(k, 0) = G_{k-1}(k, 0) = 1, \]

and for \( n \geq k \geq 2 \),

\[ G_n(k, 0) = G_{n-1}(k, 0) + G_{n-2}(k, 0) + \cdots + G_{n-k}(k, 0) \]

\[ = g_n = g_{n-1} + g_{n-2} + \cdots + g_{n-k}. \]

Let \( \mathbf{g}_n^{(k)} = (g_n^{(k)}, \ldots, g_{n-k+1}^{(k)})^T \). For any \( k \geq 2 \), the fundamental recurrence relation

\[ g_n^{(k)} = g_{n-1}^{(k)} + g_{n-2}^{(k)} + \cdots + g_{n-k}^{(k)} \]

can be defined by the vector recurrence relation \( \mathbf{g}_n^{(k)} = \mathbf{Q}_k \mathbf{g}_{n-1}^{(k)} \).

Then, we have \( \mathbf{g}_n^{(k)} = \mathbf{Q}_k^n \mathbf{g}_0^{(k)} = \mathbf{Q}_k^n (1, 1, 0, \ldots, 0)^T \). Since \( \mathbf{Q}_k \) has \( k \) distinct eigenvalues (see [5]),

\[ g_n^{(k)} = d_1 \lambda_1^n + \cdots + d_k \lambda_k^n. \]

Hence, we can determine \( d_1, d_2, \ldots, d_k \) from (8).

For example, if \( k = 3 \), then the characteristic equation of \( \mathbf{Q}_3 \) is \( f(\lambda) = \lambda^3 - \lambda^2 - \lambda - 1 = 0 \), and hence

\[
R(f(\lambda), f'(\lambda)) = \begin{vmatrix}
1 & -1 & -1 & -1 \\
0 & 1 & -1 & -1 \\
0 & 0 & 3 & -2 \\
0 & 0 & 0 & 3
\end{vmatrix} = 44 \neq 0.
\]

Thus \( f(\lambda) = 0 \) has 3 distinct roots. Suppose \( \alpha, \beta \) and \( \gamma \) are the distinct roots of \( f(\lambda) = 0 \). Then we have

\[ \alpha = \frac{1}{3} (u + v) + \frac{1}{3}, \]

\[ \beta = -\frac{1}{6} (u + v) + \frac{i \sqrt{3}}{6} (u - v) + \frac{1}{3}, \]

\[ \gamma = \frac{1}{6} (u + v) - \frac{i \sqrt{3}}{6} (u - v) + \frac{1}{3}, \]

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where
\[ i = \sqrt{-1}, \quad u = \sqrt[3]{19 + 3\sqrt{33}} \text{ and } v = \sqrt[3]{19 - 3\sqrt{33}}. \]

So, we have
\[ g_n^{(3)} = d_1\alpha^n + d_2\beta^n + d_3\gamma^n, \tag{11} \]
and hence
\[ \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}. \]

Set
\[ \delta = \det \begin{bmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{bmatrix}, \quad \delta_\alpha = \det \begin{bmatrix} 0 & 1 & 1 \\ 1 & \beta & \gamma \\ 1 & \beta^2 & \gamma^2 \end{bmatrix}, \quad \delta_\beta = \det \begin{bmatrix} 1 & 0 & 1 \\ \alpha & 1 & \gamma \\ \alpha^2 & 1 & \gamma^2 \end{bmatrix}, \]
and
\[ \delta_\lambda = \det \begin{bmatrix} 1 & 1 & 0 \\ \alpha & \beta & 1 \\ \alpha^2 & \beta^2 & 1 \end{bmatrix}. \]

Then we have
\[ d_1 = \frac{\delta_\alpha}{\delta}, \quad d_2 = \frac{\delta_\beta}{\delta}, \quad \text{and } d_3 = \frac{\delta_\gamma}{\delta}. \]

As we know, the complex numbers \(d_1, d_2,\) and \(d_3\) are independent of \(n.\)

We can also find an expression for \(g_n^{(3)}\) in [6] follows:
\[ g_n^{(3)} = \frac{(g_{n-1}^{(3)} + g_{n-2}^{(3)}) (\beta - \gamma) - (\beta^n - \alpha^n)}{(\alpha - 1)(\beta - \gamma)}. \tag{12} \]

So, by (11) and (12),
\[ \frac{\delta_\alpha \alpha^n + \delta_\beta \beta^n + \delta_\gamma \gamma^n}{\delta} = \frac{(g_{n-1}^{(3)} + g_{n-2}^{(3)}) (\beta - \gamma) - (\beta^n - \alpha^n)}{(\alpha - 1)(\beta - \gamma)}. \]
Similarly, if \( k = 2 \), then

\[
g_n^{(2)} = F_n = \frac{1}{\lambda_1 - \lambda_2} (\lambda_1^n - \lambda_2^n),
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of \( Q_2 \). Actually

\[
\lambda_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1 - \sqrt{5}}{2}.
\]

In this case,

\[
d_1 = \frac{1}{\lambda_1 - \lambda_2} = \frac{1}{\sqrt{5}}, \quad d_2 = \frac{1}{\lambda_2 - \lambda_1} = -\frac{1}{\sqrt{5}}
\]

and (13) is Binet's formula for the \( n \)th Fibonacci number \( F_n \).

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