A NON-INTEGER PROPERTY OF ELEMENTARY SYMMETRIC FUNCTIONS IN RECIPROCALs OF GENERALISED FIBONACCI NUMBERS

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1. INTRODUCTION AND MAIN RESULT

A well-known but classical result concerning the harmonic series is that the sequence of partial sums \( \sum_{r=1}^{n} \frac{1}{r} \) can never be an integer for \( n > 1 \). More generally, Nagell [3] showed that \( \sum_{r=1}^{n} \frac{1}{m+rd} \) cannot be an integer for any positive integers \( m \), \( n \) and \( d \). As an extension of these results the author, in a recent paper [4], constructed further examples of positive rational termed series having non-integer partial sums. These partial sums were of the form \( \sum_{r=1}^{n} \frac{1}{U_r} \), where \( \{U_n\} \) are the sequence of generalised Fibonacci numbers generated, for \( n \geq 2 \), via the recurrence relation

\[
U_n = PU_{n-1} - QU_{n-2},
\]

with \( U_0 = 0 \), \( U_1 = 1 \) and \((P,Q)\) a relatively prime pair of integers satisfying \(|P| > Q > 0\) or \( P \neq 0, Q < 0 \). (Note when \((P,Q) = (2,1)\) one has \( U_n = n \).) By viewing these partial sums as the symmetric function formed from summing the products of the terms \( \frac{1}{U_1}, \frac{1}{U_2}, \ldots, \frac{1}{U_n} \) taken one at a time, one may naturally ask whether all other symmetric functions in the reciprocals of such generalised Fibonacci numbers can be non-integer. In this paper we will show that for sequences \( \{U_n\} \) generated via (1), with \( P \geq 2 \) and \( Q < 0 \), there can in fact be at most finitely many \( n \) such that one or more of the elementary symmetric functions in \( \frac{1}{U_1}, \frac{1}{U_2}, \ldots, \frac{1}{U_n} \) is an integer. To establish this result we will require two preliminary Lemmas, the first of which is a refinement of Bertrand's postulate due to Ingham [2].

**Lemma 1.1:** For any real number \( x > 1 \) there always exists a prime in the interval \((x,x + \frac{1}{x})\).

The second lemma is a standard result of generalised Fibonacci sequences, a proof of which can be found in [1].

**Lemma 1.2:** For any sequence \( \{U_n\} \) generated with respect to a relatively prime pair of integers \((P,Q)\) via (1) then \((U_m, U_n) = U_{(m,n)}\).

We now can prove the following theorem:

**Theorem 1.1:** Suppose the sequence \( \{U_n\} \) is generated via (1) with respect to the relatively prime pair \((P,Q)\) such that \( P \geq 2 \) and \( Q < 0 \). Denote the \( k \)-th elementary symmetric function in \( \frac{1}{U_1}, \frac{1}{U_2}, \ldots, \frac{1}{U_n} \) by \( \phi(n,k) \), then for this family of functions there exists a uniform lower bound \( N \) on \( n \), such that \( \phi(n,k) \) is non-integer for \( n \geq N \) and \( 1 \leq k \leq n \).

**Proof:** To establish the non-integer status of \( \phi(n,k) \) it will suffice to consider the two separate cases of \( k > 3 \log n \) and \( k < 3 \log n \), noting here that it is sufficient to take only strict inequalities as \( \log n \) can never be an integer for \( n > 1 \). In both cases we will demonstrate the existence of the lower bounds given by \( N_1 = \min\{s \in \mathbb{N} : \log n \geq \frac{e^s}{3-\epsilon} \} \) and \( N_2 = \min\{s \in \mathbb{N} : \frac{9(\log n)^2}{n} + \frac{3\log n}{n} < \frac{1}{2}, \frac{n^3}{(3 \log n + 1)^3} \} > 2^8(1 + \log 3)^8 \) for

\[
n \geq s = \lfloor e^s \rfloor\text{ and } N_2 = \min\{s \in \mathbb{N} : \frac{9(\log n)^2}{n} + \frac{3\log n}{n} < \frac{1}{2}, \frac{n^3}{(3 \log n + 1)^3} \} > 2^8(1 + \log 3)^8 \text{ for }
\]
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all \( n \geq s \) respectively on \( n \), for which \( \phi(n, k) \) is non-integer. As \( N_1 \) and \( N_2 \) are constructed independently of \( k \), one can then set \( N = \max\{N_1, N_2\} \) from which it is immediate that \( \phi(n, k) \) must be non-integer for all \( n \geq N \) and \( 1 \leq k \leq n \). Furthermore, as \( N_1 \) and \( N_2 \) are not dependent on the specific choice of the sequence \( \{U_n\} \), one sees that the lower bound \( N \) must hold uniformly over the family of generalised Fibonacci sequences as specified in the theorem statement. We now proceed with the following two cases.

**Case 1: \( k > 3 \log n \)**

First note for the prescribed values of \( (P, Q) \) it can be shown, via an easy induction on \( n \), that \( U_\ast > n \). Now, as \( \phi(n, k) \) is formed from summing the terms \( \frac{1}{U_1}, \frac{1}{U_2}, \ldots, \frac{1}{U_n} \) taken \( k \) at a time, we observe that \( \phi(n, k) \) must occur \( k! \) times in the multinomial expansion \( \left( \frac{1}{U_1} + \frac{1}{U_2} + \cdots + \frac{1}{U_n} \right)^k \). Hence, using the usual comparison of \( \log n \) with the terms of the harmonic series, we obtain that

\[
\phi(n, k) < \frac{1}{k!} \left( \frac{1}{U_1} + \frac{1}{U_2} + \cdots + \frac{1}{U_n} \right)^k < \frac{1}{k!} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{n} \right)^k < \frac{1}{k!} (1 + \log n)^k. \tag{2}
\]

Now by definition of \( N_1 \) if \( n \geq N_1 \) then \( \log n > \frac{e}{3-e} \) and so \( k > \frac{3e}{3-e} \). Consequently

\[
\frac{1}{k!} (1 + \log n)^k < \frac{1}{k!} \left( 1 + \frac{k}{3} \right)^k = \frac{k^k}{k!} \left( \frac{1}{k} + \frac{1}{3} \right)^k < \left( \frac{e}{k} + \frac{e}{3} \right)^k < 1,
\]

noting here that the second last inequality follows from the fact that \( \frac{k^k}{k!} < e^k \). Hence, we deduce from the previous inequality and (2) that \( 0 < \phi(n, k) < 1 \) for any \( n \geq N_1 \) as required.

**Case 2: \( k < 3 \log n \)**

In this case it first will be necessary to show that for \( n \geq N_2 \)

\[
\left( \frac{n}{k(k+1)} - 1 \right)^5 > \left( \frac{n}{k} + 1 \right)^5. \tag{3}
\]

Upon factoring out \( \frac{n}{k} \) and \( \frac{n}{k(k+1)} \) from the right and left hand side respectively of the conjectured inequality in (3) one finds that

\[
\frac{n^3}{k^3(k+1)^5} \left( 1 - \frac{k(k+1)}{n} \right)^5 > \left( 1 + \frac{k}{n} \right)^5. \tag{4}
\]

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Now, as \( k < 3 \log n \) and so \( \frac{k}{n} < \frac{3 \log n}{n} \to 0 \) monotonically for \( n > e \), it is clear the
term \((1 + \frac{k}{n})^8\) can be bounded above by \((1 + \log 3)^5\) for \( n \geq 3 \) say. Similarly, as \( \frac{k(k+1)}{n} < \)
\( \frac{9(\log n)^2}{n} + \frac{3 \log n}{n} \to 0 \) and \( \frac{k^3 n^3}{k(k+1)^8} \to \infty \) as \( n \to \infty \), one can choose \( n \) sufficiently
large but finite and independent of \( k \), such that \( \frac{k(k+1)}{n} < \frac{1}{2} \) and \( \frac{n^3}{k(k+1)^8} > 2^8(1 + \log 3)^5 \).
Consequently by definition of \( N_2 \) one has for \( n \geq N_2 \)
\[
\frac{n^3}{k(k+1)^8} \left(1 - \frac{k(k+1)}{n}\right)^8 > (1 + \log 3)^5
\]
and so one concludes that (3) must hold for all \( n \geq N_2 \). Now raising both sides of (3) to the
power \( \frac{1}{8} \) one finds upon rearrangement that
\[
\frac{n}{k} > \left(1 + \frac{n}{k+1}\right) + \left(1 + \frac{n}{k+1}\right)^{\frac{5}{8}}.
\]

Hence for \( n \geq N_2 \) there must exist, by Lemma 1.1, a prime \( p \) in the open interval
\((1 + \frac{n}{k+1}, \frac{n}{k+1})\). By construction \( p \) must be such that \( 1 < mp < n \) for \( m = 1, 2, \ldots, k \) but
\((k+1)p > n \). Considering again \( \phi(n, k) \) as a sum of the product of the terms \( \frac{1}{U_1}, \frac{1}{U_2}, \ldots, \frac{1}{U_n} \)
taken \( k \) at a time we can write
\[
\phi(n, k) = \left(\begin{array}{c}n \\ k\end{array}\right) \sum_{i=1}^{\left(\begin{array}{c}n \\ k\end{array}\right)} \frac{1}{c_i} = \frac{b_1 + b_2 + \cdots + b_{\left(\begin{array}{c}n \\ k\end{array}\right)}}{U_1 U_2 \ldots U_n} = \frac{B}{C},
\]
where \( c_i \) is one of the possible \( \left(\begin{array}{c}n \\ k\end{array}\right) \) products of the terms \( U_1, U_2, \ldots, U_n \) taken \( k \) at a time and
\[
b_i = \frac{U_1 U_2 \ldots U_n}{c_i}.
\]
By the above \( U_p U_2p \ldots U_{kp} = c_s \), for some \( s \in \{1, 2, \ldots, \left(\begin{array}{c}n \\ k\end{array}\right)\} \), and as \((k+1)p > n \), no other of
the remaining \( \left(\begin{array}{c}n \\ k\end{array}\right) - 1 \) products \( c_i \) can contain generalised Fibonacci numbers in which all of
the corresponding \( k \) subscripts are a multiple of \( p \). Consequently, by construction each \( b_i \), with

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$i \neq s$, must contain at least one of the terms in the set $A = \{U_p, U_{2p}, \ldots, U_{kp}\}$ while $b_s$ will contain none of the terms in $A$. Now by Lemma 1.2 as $p$ is prime $(U_p, U_{mp}) = U_{(p, mp)} = U_p$, for each $m = 1, 2, \ldots, k$, and so $U_p|b_i$ for every $i \neq s$. Also for $(r, p) = 1$ one has $(U_p, U_r) = U_1 = 1$ but as $b_s$ contains only those terms $U_r$ for which $(r, p) = 1$, we conclude that $U_p$ must be relatively prime to $b_s$, and so $U_p|b_s$, which in turn implies that $U_p|B$. Thus $\phi(n, k) = \frac{R}{C}$ where $U_p|C$ but $U_p|B$, that is $\phi(n, k)$ cannot be an integer for any $n \geq N_2$ as required. □

Remark 1.1: It is clear that the above argument could easily be applied to higher order recurrences $\{U_n\}$ with $U_n \geq n$ if an analogous result in Lemma 1.2 could be found.

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REFERENCES


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