A PROBABILISTIC VIEW OF CERTAIN WEIGHTED FIBONACCI SUMS

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(Submitted May 2001)

1. INTRODUCTION

In this paper we investigate sums of the form

\[ a_n := \sum_{k \geq 1} \frac{k^n F_k}{2^{k+1}}. \]  

(1)

For any given \( n \), such a sum can be determined [3] by applying the \( x \frac{d}{dx} \) operator \( n \) times to the generating function

\[ G(x) := \sum_{k \geq 1} F_k x^k = \frac{x}{1 - x - x^2}, \]

then evaluating the resulting expression at \( x = 1/2 \). This leads to \( a_0 = 1, a_1 = 5, a_2 = 47 \), and so on. These sums may be used to determine the expected value and higher moments of the number of flips needed of a fair coin until two consecutive heads appear [3]. In this article, we pursue the reverse strategy of using probability to derive \( a_n \) and develop an exponential generating function for \( a_n \) in Section 3. In Section 4, we present a method for finding an exact, non-recursive, formula for \( a_n \).

2. PROBABILISTIC INTERPRETATION

Consider an infinitely long binary sequence of independent random variables \( b_1, b_2, b_3, \ldots \) where \( P(b_i = 0) = P(b_i = 1) = 1/2 \). Let \( Y \) denote the random variable denoting the beginning of the first 00 substring. That is, \( b_Y = b_{Y+1} = 0 \) and no 00 occurs before then. Thus \( P(Y = 1) = 1/4 \). For \( k \geq 2 \), we have \( P(Y = k) \) is equal to the probability that our sequence begins \( b_1, b_2, \ldots, b_{k-2}, 1, 0, 0 \), where no 00 occurs among the first \( k - 2 \) terms. Since
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the probability of occurrence of each such string is \((1/2)^{k+1}\), and it is well known [1] that there are exactly \(F_k\) binary strings of length \(k \geq 2\) with no consecutive 0’s, we have for \(k \geq 1\),

\[
P(Y = k) = \frac{F_k}{2^{k+1}}.
\]

Since \(Y\) is finite with probability 1, it follows that

\[
\sum_{k \geq 1} \frac{F_k}{2^{k+1}} = \sum_{k \geq 1} P(Y = k) = 1.
\]

For \(n \geq 0\), the expected value of \(Y^n\) is

\[
a_n := E(Y^n) = \sum_{k \geq 1} k^n F_k \frac{1}{2^{k+1}}.
\]

Thus \(a_0 = 1\). For \(n \geq 1\), we use conditional expectation to find a recursive formula for \(a_n\). We illustrate our argument with \(n = 1\) and \(n = 2\) before proceeding with the general case.

For a random sequence \(b_1, b_2, \ldots,\), we compute \(E(Y)\) by conditioning on \(b_1\) and \(b_2\). If \(b_1 = b_2 = 0\), then \(Y = 1\). If \(b_1 = 1\), then we have wasted a flip, and we are back to the drawing board; let \(Y’\) denote the number of remaining flips needed. If \(b_1 = 0\) and \(b_2 = 1\), then we have wasted two flips, and we are back to the drawing board; let \(Y”\) denote the number of remaining flips needed in this case. Now by conditional expectation we have

\[
E(Y) = \frac{1}{4}(1) + \frac{1}{2}E(1 + Y’) + \frac{1}{4}E(2 + Y”)
\]

\[
= \frac{1}{4} + \frac{1}{2} + \frac{1}{2}E(Y’) + \frac{1}{2} + \frac{1}{4}E(Y”)
\]

\[
= \frac{5}{4} + \frac{3}{4}E(Y)
\]

since \(E(Y’) = E(Y”) = E(Y)\). Solving for \(E(Y)\) gives us \(E(Y) = 5\). Hence,

\[
a_1 = \sum_{k \geq 1} k F_k \frac{1}{2^{k+1}} = 5.
\]

Conditioning on the first two outcomes again allows us to compute

\[
E(Y^2) = \frac{1}{4}(1^2) + \frac{1}{2}E[(1 + Y’)^2] + \frac{1}{4}E[(2 + Y”)^2]
\]

\[
= \frac{1}{4} + \frac{1}{2}E(1 + 2Y + Y^2) + \frac{1}{4}E(4 + 4Y + Y^2)
\]

\[
= \frac{7}{4} + 2E(Y) + \frac{3}{4}E(Y^2).
\]

Since \(E(Y) = 5\), it follows that \(E(Y^2) = 47\). Thus,

\[
a_2 = \sum_{k \geq 1} k^2 F_k \frac{1}{2^{k+1}} = 47.
\]
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Following the same logic for higher moments, we derive for \( n \geq 1 \),

\[
E(Y^n) = \frac{1}{4} (1^n) + \frac{1}{2} E[(1 + Y)^n] + \frac{1}{4} E[(2 + Y)^n]
\]

\[
= \frac{1}{4} + \frac{3}{4} E(Y^n) + \frac{1}{2} \sum_{k=0}^{n-1} \binom{n}{k} E(Y^k) + \frac{1}{4} \sum_{k=0}^{n-1} \binom{n}{k} 2^{n-k} E(Y^k).
\]

Consequently, we have the following recursive equation:

\[
E(Y^n) = 1 + \sum_{k=0}^{n-1} \binom{n}{k} [2 + 2^{n-k}] E(Y^k)
\]

Thus for all \( n \geq 1 \),

\[
a_n = 1 + \sum_{k=0}^{n-1} \binom{n}{k} [2 + 2^{n-k}] a_k.
\] (3)

Using equation (3), one can easily derive \( a_3 = 665, a_4 = 12,551 \), and so on.

3. GENERATING FUNCTION AND ASYMPTOTICS

For \( n \geq 0 \), define the exponential generating function

\[
a(x) = \sum_{n \geq 0} a_n \frac{x^n}{n!}.
\]

It follows from equation (3) that

\[
a(x) = 1 + \sum_{n \geq 1} \frac{\left(1 + \sum_{k=0}^{n-1} \binom{n}{k} [2 + 2^{n-k}] a_k\right)}{n!} x^n
\]

\[
= e^x + 2a(x)(e^x - 1) + a(x)(e^{2x} - 1).
\]

Consequently,

\[
a(x) = \frac{e^x}{4 - 2e^x - e^{2x}}.
\] (4)

For the asymptotic growth of \( a_n \), one need only look at the leading term of the Laurent series expansion [4] of \( a(x) \). This leads to

\[
a_n \approx \frac{\sqrt{5} - 1}{10 - 2\sqrt{5}} \left( \frac{1}{\ln(\sqrt{5} - 1)} \right)^{n+1} n!.
\] (5)

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4. CLOSED FORM

While the recurrence (3), generating function (4), and asymptotic result (5) are satisfying, a closed form for \( a_n \) might also be desired. For the sake of completeness, we demonstrate such a closed form here.

To calculate

\[
a_n = \sum_{k \geq 1} \frac{k^n F_k}{2^{k+1}},
\]

we first recall the Binet formula for \( F_k \) [3]:

\[
F_k = \frac{1}{\sqrt{5}} \left( \left( \frac{1 + \sqrt{5}}{2} \right)^k - \left( \frac{1 - \sqrt{5}}{2} \right)^k \right)
\]  

(6)

Then (6) implies that (1) can be rewritten as

\[
a_n = \frac{1}{2\sqrt{5}} \sum_{k \geq 1} k^n \left( \frac{1 + \sqrt{5}}{4} \right)^k - \left( \frac{1 - \sqrt{5}}{4} \right)^k.
\]  

(7)

Next, we remember the formula for the geometric series:

\[
\sum_{k \geq 0} x^k = \frac{1}{1 - x}
\]  

(8)

This holds for all real numbers \( x \) such that \( |x| < 1 \). We now apply the \( \frac{d}{dx} \) operator \( n \) times to (8). It is clear that the left-hand side of (8) will then become

\[
\sum_{k \geq 1} k^n x^k.
\]

The right-hand side of (8) is transformed into the rational function

\[
\frac{1}{(1 - x)^{n+1}} \times \sum_{j=1}^{n} e(n, j) x^j,
\]  

(9)

where the coefficients \( e(n, j) \) are the Eulerian numbers [2, Sequence A008292], defined by

\[
e(n, j) = j \cdot e(n - 1, j) + (n - j + 1) \cdot e(n - 1, j - 1) \quad \text{with} \quad e(1, 1) = 1.
\]

(The fact that these are indeed the coefficients of the polynomial in the numerator of (9) can be proved quickly by induction.) From the information found in [2, Sequence A008292], we know

\[
e(n, j) = \sum_{\ell=0}^{j} (-1)^\ell (j - \ell)^n \binom{n + 1}{\ell}.
\]
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Therefore,

\[ \sum_{k \geq 1} k^n x^k = \frac{1}{(1-x)^{n+1}} \times \sum_{j=1}^{n} \left[ \sum_{\ell=0}^{j} (-1)^\ell (j-\ell)^n \binom{n+1}{\ell} \right] x^j. \] \hspace{1cm} (10)

Thus the two sums

\[ \sum_{k \geq 1} k^n \left( \frac{1 + \sqrt{5}}{4} \right)^k \text{ and } \sum_{k \geq 1} k^n \left( \frac{1 - \sqrt{5}}{4} \right)^k \]

that appear in (7) can be determined explicitly using (10) since

\[ \left| \frac{1 + \sqrt{5}}{4} \right| < 1 \text{ and } \left| \frac{1 - \sqrt{5}}{4} \right| < 1. \]

Hence, an exact, non-recursive, formula for \(a_n\) can be developed.

REFERENCES


AMS Classification Numbers: 11B39

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