DOUBLE INDEXED FIBONACCI SEQUENCES AND THE BIVARIATE PROBABILITY DISTRIBUTION

Belkheir Essebbar

Department of Mathematics and Computer Science

Faculté des Sciences of Rabat, Morocco

essebbar@fsr.ac.ma

(Submitted July 2000-Final Revision May 2003)

1. INTRODUCTION

The classical r-Fibonacci sequence $(U_i)_{i\geq 0}$ is defined by some given real numbers $U_0, U_1, \ldots, U_{r-1}$ and the difference equation

$$\sum_{k=0}^{r} a_k U_{i-k} = 0; \quad i \geq r,$$

where $a_k, k = 0, 1, \ldots, r$ are arbitrary real numbers such that $a_r \neq 0, r \geq 2$. The characteristic polynomial of this equation is given by

$$Q(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_r x^r.$$

Many authors have studied the r-Fibonacci sequence given above, see for example Mouline and Rachidi (1998). Philippou et. al. (1982) and Philippou (1988) have related the Fibonacci sequences to the one dimensional geometric probability distribution.

Now we introduce the double indexed Fibonacci sequence (DIFS) of order (n, m). Let $(U_{ij})_{i\geq 0, j\geq 0}$ be the double indexed sequence defined by the difference equations of order (n, m) :

$$\sum_{k_1=0}^{n} \sum_{k_2=0}^{m} a_{k_1,k_2} U_{i-k_1, j-k_2} = 0, \quad i \geq n, j \geq m, \tag{1}$$

where $a_{i,j}, i = 0, \ldots, n, j = 0, \ldots, m$ are real numbers such that $a_{00} \neq 0$ and $a_{n,m} \neq 0$. The corresponding characteristic polynomial is defined by

$$Q(x, y) = a_{00} + a_{10} x + a_{01} y + a_{11} xy + \cdots + a_{nm} x^n y^m.$$

Next, we recall that (X, Y) is a discrete random vector in two dimensions with values in $N \times N$ and defined on a certain underlying probability space (Ω, A, P) with probability generating function given by

$$g(x, y) = E(x^X y^Y) = \sum_{i\geq 0} \sum_{j\geq 0} p_{ij} x^i y^j,$$

where $p_{ij} = P(X = i, Y = j), i \geq 0, j \geq 0$ is the probability mass function of (X, Y). For example, the bivariate negative binomial distribution has the probability generating function defined by

$$g(x,y) = \left(\frac{d}{1 - ax - by - cxy}\right)^r, \tag{2}$$

where a, b, c and d are real numbers such that $0 \leq a \leq 1, 0 \leq b \leq 1, 0 < d < 1$ and $a+b+c+d = 1$. For $r = 1$, we get the bivariate geometric distribution as a special case of (2). For more details about these distributions and their applications see Edwards et. al. (1961), Feller (1968) (page 285), Subrahmanian et. al. (1973) and Davy et. al. (1996).

Philippou et. al. (1989, 1990, 1991) and Antzoulakos et. al. (1991) have related the special case of the above distribution when the crossed term in x and y is null ($c = 0$) with extensions to some particular characteristic polynomials (see section 3).

This work is organized as follows: in the second section we develop the setting of the difference equation given by (1). In section 3, we give examples of DIFS with their combinatorial solutions. In section 4, we study the relationships of DIFS and the bivariate probability distributions given by (2).

2. THE DIFS

Let $(U_{ij})_{i\geq 0, j\geq 0}$ be the DIFS of order (n, m) as given by (1), that is,

$$\sum_{k_1=0}^{n} \sum_{k_2=0}^{m} a_{k_1,k_2} U_{i-k_1,j-k_2} = 0, \quad i \geq n, j \geq m, \tag{3}$$

with the initial conditions: $a_{00}U_{00} = 1$ and $a_{n,m} \neq 0$ with $U_{ij} = 0$ if $i < 0$ or $j < 0$. Now let $Q(x, y)$ be the corresponding characteristic polynomial with order (n, m) of the difference equation given by (3), that is,

$$Q(x,y) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{m} a_{k_1,k_2} x^{k_1} y^{k_2}, \tag{4}$$

with $a_{0,0} \neq 0$ and $a_{n,m} \neq 0$. Then we have the development of $1/Q(x, y)$ using power series, that is,

$$\frac{1}{Q(x,y)} = \sum_{i\geq 0, j\geq 0} U_{i,j} x^i y^j.$$

From the equality $Q(x,y)/Q(x,y) = 1$, we get

$$\sum_{k_1=0}^{n}\sum_{k_2=0}^{m} a_{k_1,k_2}x^{k_1}y^{k_2} \cdot \sum_{i\geq 0, j\geq 0} U_{i,j}x^i y^j = 1,$$

which gives the difference equations defined by (3) with specific initial conditions. The combinatorial solution of (3), $(U_{ij})_{i\geq 0, j\geq 0}$, is then given by means of the development of $1/Q(x,y)$ using power series.

There are two cases which the combinatorial solution of (3) could be deduced from elementary combinatorial solutions. Let $Q(x,y)$ be a characteristic polynomial of order (n,m) such that the following decomposition holds

$$Q(x,y) = \prod_{k=1}^{r} Q_k(x,y), \tag{5}$$

where the order of each polynomial $Q_k(x,y)$ is (n_k, m_k), $k = 1,\ldots,r$ with $n = n_1 + \cdots + n_r$ and $m = m_1 + \cdots + m_r$. Let $U_{i,j}^{(k)}$ be the combinatorial solution of $Q_k(x,y)$, that is,

$$\frac{1}{Q_k(x,y)} = \sum_{i,j} U_{i,j}^{(k)} x^i y^j, \quad k = 1,\ldots,r.$$

Let us establish the following result which gives the convolution of independent DIFS's.

Theorem 1: *If* $(U_{i,j})$ *is the combinatorial solution of* (3) *with* $Q(x,y)$ *verifying the decomposition* (5) *then*

$$U_{ij} = \sum_{i_1=0}^{i}\sum_{i_2=0}^{i-i_1}\cdots\sum_{i_{r-1}=0}^{i-(i_1+\cdots+i_{r-2})}\sum_{j_1=0}^{j}\sum_{j_2=0}^{j-j_1}\cdots\sum_{j_{r-1}=0}^{j-(j_1+\cdots+j_{r-2})}$$

$$U_{i_1,j_1}^{(1)}U_{i_2,j_2}^{(2)}\cdots U_{i_{r-1},j_{r-1}}^{(r-1)}U_{i-(i_1+\cdots+i_{r-1}),j-(j_1+\cdots+j_{r-1})}^{(r)}.$$

Proof: One has,

$$\frac{1}{Q(x,y)} = \prod_{k=1}^{r}\frac{1}{Q_k(x,y)}$$

$$= \sum_{i_1,\ldots,i_r}\sum_{j_1,\ldots,j_r} U_{i_1,j_1}^{(1)}U_{i_2,j_2}^{(2)}\cdots U_{i_r,j_r}^{(r)} x^{i_1+\cdots+i_r}y^{j_1+\cdots+j_r}.$$

Set $i = i_1 + \cdots + i_r$ and $j = j_1 + \cdots + j_r$. Then

$$\frac{1}{Q(x,y)} = \sum_{i,j} x^i y^j \sum_{i_1=0}^{i} \sum_{i_2=0}^{i-i_1} \cdots \sum_{i_{r-1}=0}^{i-(i_1+\cdots+i_{r-2})} \sum_{j_1=0}^{j} \sum_{j_2=0}^{j-j_1} \cdots \sum_{j_{r-1}=0}^{j-(j_1+\cdots+j_{r-2})}$$

$$U^{(1)}_{i_1,j_1} U^{(2)}_{i_2,j_2} \cdots U^{(r-1)}_{i_{r-1},j_{r-1}} U^{(r)}_{i-(i_1+\cdots+i_{r-1}),j-(j_1+\cdots+j_{r-1})}.$$

By the identification with

$$\frac{1}{Q(x,y)} = \sum_{i,j} U_{i,j} x^i y^j$$

one can deduce the result of the theorem. □

Let us now suppose that the characteristic polynomial, $Q(x,y)$, of equation (3) can be decomposed as

$$Q(x,y) = Q_1(x) \cdot Q_2(y) \tag{6}$$

where $Q_1(x)$ and $Q_2(y)$ are polynomials with respective orders n and m such that

$$\frac{1}{Q_1(x)} = \sum_i U^{(1)}_i x^i, \quad \frac{1}{Q_2(y)} = \sum_j U^{(2)}_j y^j.$$

Let us establish this result.

Theorem 2: *The characteristic polynomial $Q(x,y)$ of (3) is decomposed as in (6) if and only if the combinatorial solution is given by*

$$U_{ij} = U^{(1)}_i U^{(2)}_j$$

Proof: First

$$\frac{1}{Q(x,y)} = \frac{1}{Q_1(x)} \cdot \frac{1}{Q_2(y)}$$

$$= \sum_{i,j} U^{(1)}_i U^{(2)}_j x^i y^j.$$

Then, by identification with

$$\frac{1}{Q(x,y)} = \sum_{i,j} U_{i,j} x^i y^j.$$

the result follows. $\square$

Let us introduce the notion of marginals of the DIFS. Let U_{ij} be a DIFS of order (n, m) as given by (3) and $Q(x, y)$ be the corresponding characteristic polynomial as given by (4). We define the marginal polynomials as polynomials in x or in y, that is, $Q(x, 1)$ and $Q(1, y)$ given by

$$Q(x,1) = \sum_{l,k} a_{lk} x^l = \sum_{l=0}^{n} x^l \sum_{k=0}^{m} a_{lk},$$

$$Q(1,y) = \sum_{l,k} a_{lk} y^k = \sum_{k=0}^{m} y^k \sum_{l=0}^{n} a_{lk}.$$

The associated equations are respectively

$$\sum_{l=0}^{n} V_{i-l} \sum_{k=0}^{m} a_{lk} = 0,$$

$$\sum_{k=0}^{m} W_{j-k} \sum_{l=0}^{n} a_{lk} = 0,$$

with $V_i = \sum_j U_{ij}$ and $W_j = \sum_i U_{ij}$ which are the combinatorial marginal solutions.

3. EXAMPLES

(a) DIFS of order (1,1): Let U_{ij} be the DIFS given by (3) with $n = m = 1$, that is,

$$\sum_{k_1=0}^{1} \sum_{k_2=0}^{1} a_{k_1,k_2} U_{i-k_1,j-k_2} = 0.$$

This is equivalent to

$$a_{00}U_{i,j} + a_{10}U_{i-1,j} + a_{01}U_{i,j-1} + a_{11}U_{i-1,j-1} = 0, \tag{7}$$

with $a_{00}U_{00} = 1$. The associated characteristic polynomial is then

$$Q(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy.$$

For the developement of $1/Q(x, y)$ using power series, we first need to establish the following lemma.

Lemma 1: *Let a, b, c be real numbers and r be a positive integer. One has,*
(i) if $abc \neq 0$ and (x, y) is such that $|ax + by + cxy| < 1$, then

$$\frac{1}{(1 - ax - by - cxy)^r} = \sum_{i,j} x^i y^j a^i b^j \sum_{k=0}^{\min(i,j)} \left(\frac{c}{ab}\right)^k C_i^k C_{i+j-k}^i C_{i+j+r-k-1}^{r-1},$$

(ii) if $a = 0, bc \neq 0$ and (x, y) is such that $|by + cxy| < 1$, then

$$\frac{1}{(1 - by - cxy)^r} = \sum_{i,j} (by)^i (cxy)^j C_{i+j}^i C_{i+j+r-1}^{r-1},$$

(iii) if $b = 0, ac \neq 0$ and (x, y) is such that $|ax + cxy| < 1$, then

$$\frac{1}{(1 - ax - cxy)^r} = \sum_{i,j} (ax)^i (cxy)^j C_{i+j}^i C_{i+j+r-1}^{r-1},$$

(iv) if $ab \neq 0, c = 0$ and (x, y) is such that $|ax + by| < 1$, then

$$\frac{1}{(1 - ax - by)^r} = \sum_{i,j} (ax)^i (by)^j C_{i+j}^i C_{i+j+r-1}^{r-1},$$

where $C_n^i = n!i!/(n-i)!$ and the summation $\sum_{i,j}$ is over $i \geq 0, j \geq 0$.

Proof:
(i) It is known that for $|t| < 1$ the expansion of $1/(1-t)^r$ is $\sum_{n \geq 0} t^n C_{n+r-1}^{r-1}$. For $t = ax + by + cxy$, one has,

$$\frac{1}{(1 - ax - by - cxy)^r} = \sum_{n \geq 0} (ax + by + cxy)^n C_{n+r-1}^{r-1}.$$

Now by the multinomial formula, one has,

$$(ax + by + cxy)^n = \sum_{k_1+k_2+k_3=n} (ax)^{k_1} (by)^{k_2} (cxy)^{k_3} \frac{n!}{k_1!k_2!k_3!}$$

$$= \sum_{k_1+k_2+k_3=n} a^{k_1} b^{k_2} c^{k_3} x^{k_1+k_3} y^{k_2+k_3} \frac{n!}{k_1!k_2!k_3!},$$

with the conventions that $0^0 = 0! = 1$. Let us put $i = k_1 + k_3$ and $j = k_2 + k_3$. One can see that $i = 0, \dots, n, j = 0, \dots, n$ and $0 \leq k_3 \leq i, 0 \leq k_3 \leq j$, that is $0 \leq k_3 \leq min(i,j)$. Then for $ab \neq 0$

$$\frac{1}{(1-ax-by-cxy)^r} = \sum_{n\geq 0} C_{n+r-1}^{r-1} \sum_{i+j-k_3=n} (ax)^i (by)^j \left(\frac{c}{ab}\right)^{k_3} \frac{n!}{(i-k_3)!(j-k_3)!k_3!}$$

$$= \sum_{i>0} (ax)^i \sum_{j\geq 0} (by)^j \sum_{k_3=0}^{min(i,j)} \left(\frac{c}{ab}\right)^{k_3} \frac{(i+j-k_3)!}{(i-k_3)!(j-k_3)!k_3!} C_{i+j+r-k_3-1}^{r-1},$$

which coincides with the claimed formula.

(ii) If $a = 0$ and $bc \neq 0$, one has

$$(by + cxy)^n = \sum_{k=0}^{n} (by)^k (cxy)^{n-k} C_n^k.$$

Then

$$\frac{1}{(1-by-cxy)^r} = \sum_{n\geq 0} C_{n+r-1}^{r-1} \sum_{k=0}^{n} (by)^k (cxy)^{n-k} C_n^k$$

$$= \sum_{k\geq 0} (by)^k \sum_{j\geq 0} (cxy)^j C_{k+j}^k C_{k+j+r-1}^{r-1}.$$

One can easily derive the other expressions of the lemma. □

Let us derive from Lemma 1. the expression of $1/Q(x,y)$ using power series. It is easy to see that with the parameterizations $a_{00} = 1/d, a_{10} = -a/d, a_{01} = -b/d, a_{1,1} = -c/d$, (7) becomes

$$U_{ij} = aU_{i-1,j} + bU_{i,j-1} + cU_{i-1,j-1}, \quad i \geq 1, j \geq 1, \tag{8}$$

with the initial conditions: $U_{00} = d, U_{10} = ad$, and $U_{01} = bd$. The associated characteristic polynomial is

$$Q(x,y) = \frac{1-ax-by-cxy}{d}.$$

From the above Lemma 1 and for $r = 1$ (with for example $ab \neq 0$) one has

$$\frac{1}{Q(x,y)} = \sum_{i,j} U_{ij} x^i y^j$$

with

$$U_{ij} = da^i b^j \sum_{k=0}^{min(i,j)} \left(\frac{c}{ab}\right)^k \frac{(i+j-k)!}{(i-k)!(j-k)!k!}$$

$$= da^i b^j \sum_{k=0}^{min(i,j)} \left(\frac{c}{ab}\right)^k C_i^k C_{i+j-k}^i,$$

which is the combinatorial solution of equation (8). For $c = 0$, which means that the crossed term is null, the solution is then $U_{ij} = da^i b^j C_{i+j}^i$. This is the solution of the m-variate generalized Fibonacci polynomial of order k (here $m = 2$, $k = 1$) given by Philippou et. al. (1991) and Antzoulakos et. al. (1991).

(b) DIFS of order (r, r): Let Q be

$$Q(x,y) = (1 - ax - by - cxy)^r / d^r,$$

which is the characteristic polynomial of the DIFS of order (r, r) given by

$$\sum_{k_1+k_2+k_3+k_4=r} (-a)^{k_2}(-b)^{k_3}(-c)^{k_4} \frac{r!}{k_1!k_2!k_3!k_4!} U^{(r)}_{i-(k_2+k_4),j-(k_3+k_4)} = 0. \tag{9}$$

This is equivalent to

$$\sum_{k_1=0}^{r} C_r^{k_1} \sum_{k_2=0}^{r-k_1} (-a)^{k_2} C_{r-k1}^{k_2} \sum_{k_3=0}^{r-k_1-k_2} (-b)^{k_3}(-c)^{r-k_1-k_2-k_3}$$

$$C_{r-k_1-k_2}^{k_3} U^{(r)}_{i-r+k_1+k_3,j-r+k_1+k_2} = 0.$$

From Lemma 1, one can deduce the combinatorial solution of the above equation. That is,

$$U_{ij}^{(r)} = d^r a^i b^j \sum_{k=0}^{min(i,j)} \left(\frac{c}{ab}\right)^k C_i^k C_{i+j-k}^i C_{i+j+r-k-1}^{r-1}, \quad i \geq 0, j \geq 0. \tag{10}$$

This solution can also be derived from Theorem 1 since the characteristic polynomial has the decomposition (5). Since the DIFS of order (r, r) is the r-fold convolution of DIFS of order

(1,1), Philippou et. al. (1991) showed that the solution can be evaluated (for small values of r) recursively as

$$U_{ij}^{(r)} = \sum_{k_1=0}^{i} \sum_{k_2=0}^{j} U_{k_1,k_2} U_{i-k_1,j-k_2}^{(r-1)}, \quad r \geq 2,$$

with $U_{ij}^{(1)} = U_{ij}$ which is the solution of DIFS of order (1,1).

4. THE DIFS AND THE BIVARIATE PROBABILITY DISTRIBUTION

A random couple (X, Y) has a bivariate negative binomial distribution if its probability generating function has the form given by (2), that is

$$g(x,y) = \left(\frac{d}{1 - ax - by - cxy}\right)^r,$$

with a, b, c and d such that $0 \leq a < 1, 0 \leq b < 1, 0 < d < 1$ and $a + b + c + d = 1$. This distribution is the convolution of r independent bivariate geometric distributions. One can recognize the associated stochastic DIFS given in example (b) which the combinatorial solution given by (10) is the probability mass function associated to this bivariate negative binomial distribution, that is for $i \geq 0, j \geq 0$,

$$P(X = i, Y = j) = d^r a^i b^j \sum_{k=0}^{min(i,j)} \left(\frac{c}{ab}\right)^k C_i^k C_{i+j-k}^i C_{i+j+r-k-1}^{r-1}. \tag{11}$$

Equation (9) permits recursive computations of the above probabilities which are more convenient than those given by Edwards et. al. (1961) and Subrahmanian et. al. (1973). The geometric case ($r = 1$) is given by equation (8).

The random variables X and Y have the marginal distributions which are negative binomial distributions with respectively the parameters $(r, d/(1-b))$ and $(r, d/(1-a))$, that is

$$P(X = i) = \left(\frac{d}{1-b}\right)^r U_i; \quad i \geq 0, \tag{12}$$

where (U_i) is the marginal Fibonacci sequence satisfying

$$\sum_{k=0}^{r} C_r^k \left(\frac{a+c}{b-1}\right)^k U_{i-k} = 0, \tag{13}$$

with $U_0 = 1$. The same holds with $P(Y = j)$. The relations (12) and (13) are less practical for the evaluation of $P(X = i)$, since the following recursive scheme is more effective

$$P(X = i) = q\left(1 + \frac{r-1}{i}\right) P(X = i - 1); \quad i \geq 1,$$

starting with the initial condition $P(X = 0) = p^r$, where $p = d/(1 - b)$ and $p + q = 1$.

The covariance of X and Y is given by (see for example Subrahmanian et. al. (1973))

$$Cov(X, Y) = r\frac{c + ab}{d^2}.$$

It is easy to see that X and Y are independent random variables if and only if $Cov(X, Y) = 0 (c = -ab)$. We can express this result as follows.

Lemma 2: *For $i \geq 0, j \geq 0$ and $r \geq 1$, one has*

$$C^{r-1}_{r+i-1} C^{r-1}_{r+j-1} = \sum_{k=0}^{min(i,j)} (-1)^k C^k_i C^i_{i+j-k} C^{r-1}_{i+j+r-k-1}.$$

Proof: Let $g(x, y) = d^r/(1 - ax - by - cxy)^r$. The independence of X and Y is given by $c = -ab$, that is

$$\begin{aligned} g(x,y) &= \left(\frac{1-a}{1-ax}\right)^r \left(\frac{1-b}{1-by}\right)^r \\ &= (1-a)^r \sum_{i\geq 0} (ax)^i C^{r-1}_{r+i-1} \cdot (1-b)^r \sum_{j \geq 0} (by)^j C^{r-1}_{r+j-1} \\ &= \sum_{i,j} U_{ij} x^i y^j \end{aligned}$$

with

$$U_{ij} = (1-a)^r (1-b)^r a^i b^j C^{r-1}_{r+i-1} C^{r-1}_{r+j-1}.$$

By identification with (11) when $c = -ab$, we get the result of the lemma. □

5. CONCLUSION

As in the case of the one indexed Fibonacci sequences, there is a relationship between the probability distributions of discrete type and these sequences. In this work, some basic definitions, results and examples of DIFS are given. The link between stochastic DIFS and the bivariate negative binomial distribution is established.

There are plenty of problems left to be solved such as: the combinatorial solution of (1) with arbitrary real numbers (a_{ij}), the roots of the characteristic polynomial.

Also the generalization to the multiple indexed Fibonacci sequences and their relationship with multivariate probability distribution opens the door to a host of other quesitons.

ACKNOWLEDGMENT

The author is deeply grateful to an anonymous referee for his advice which improved the quality of this work. Research grants from CNR under project PROTARS P3T1/06 are also gratefully acknowledged.

REFERENCES

[1] D.L. Antzoulakos and A.N. Philippou. "A Note on the Multivariate Negative Binomial Distributions of Order k." *Communications in Statistics - Theory and Methods* **20** (1991): 1389-1399.

[2] P.J. Davy and J.C.W. Ranger. "Multivariate Geometric Distributions." *Comm. in Stat., Theory and Methods* **25** (1996): 2971-2987.

[3] C.B. Edwards and J. Gurland. "A Class of Distributions Applicable to Accidents." *JASA* **56** (1961): 503-517.

[4] W. Feller. *An Introduction to Probability Theory and Its Applications.* Volume 1, Third edition, New-York: Willey, 1968.

[5] M. Mouline and M. Rachidi. "Application of Markov Chains to r-Generalized Fibonacci Sequences." *The Fibonacci Quarterly* **37** (1998): 34-38.

[6] G.N. Philippou. "On the k-th Order Linear Recurrence and Some Probability Applications." *Applications of Fibonacci Numbers*, Kluwer Academic Publishers, (1988): 89-96.

[7] A.N. Philippou and A.A. Muwafi. "Waiting for the k-th Consecutive Success and the Fibonacci Sequence of Order k." *The Fibonacci Quarterly* **20** (1982): 28-32.

[8] A.N. Philippou, D.L. Antzoulakos and G.A. Tripsiannis. "Multivariate Distributions of Order k." *Statistics and Probability Letters* **7** (1989): 207-216.

[9] A.N. Philippou and D.L. Antzoulakos. "Multivariate Distributions of Order k on a Generalized Sequence." *Statistics and Probability Letters* **9** (1990): 453-463.

[10] A.N. Philippou and D.L. Antzoulakos. "Generalized Multivariate Fibonacci Polynomials of Order k and the Multivariate Negative Binomial Distributions of the Same Order." *The Fibonacci Quarterly* **29** (1991): 322-328.

[11] K. Subrahmanian, and K. Subrahmanian. "On the Estimation of the Parameters in the Bivariate Negative Binomial Distribution." *Journal of the Roy. Stat. Soc. B* **35** (1973): 131-146.

AMS Classification Numbers: 40A05, 40A25, 45M05

✠ ✠ ✠