1. INTRODUCTION

Let \( r > 0 \) be a fixed real number. In this paper we will study infinite series of the form:

\[
x = \sum_{n=1}^{\infty} \frac{\epsilon_n}{(F_n)^r} \quad (\epsilon_n = 1 \text{ or } 0),
\]

where \( x \in [0, I_r] \). \( I_r \) signifies the sum of series (1) if \( \epsilon_n = 1 \) for all \( n \in N \). The convergence of the series (1), if \( x = I_r \), can be easily proved by the well-known Binet formula! Letting \( \alpha = (1 + \sqrt{5})/2 \) and \( \beta = (1 - \sqrt{5})/2 \) we have

\[
F_n = (\alpha^n - \beta^n) / \sqrt{5}.
\]

Notice that \( 0 < \alpha^{-r} < 1 \) and that Binet’s formula yields \( \lim_{n \to \infty} (F_n)^r/\alpha^{rn} = (\sqrt{5})^{-r} \). Thus applying the quotient-criterion for infinite series and geometric series proves the convergence of (1). For example: \( I_1 = 3,359 \ldots \) Furthermore it is easy to see that

\[
I_r > I_{r'} \text{ for } r < r', \quad I_r \to \infty \text{ for } r \to 0 \text{ and } I_r \to 2 \text{ for } r \to \infty.
\]

We begin with certain results due to J.L. Brown in [1] and P. Ribenboim in [8] dealing with the representation of real numbers in the form (1). In [1] J.L. Brown treated the case \( r = 1 \). In [8] P. Ribenboim proved that for every positive real number \( x \) there exists a unique integer \( m \geq 1 \) such that \( I_{1/(m-1)} < x \leq I_{1/m} \) and \( x \) is representable in the form (1) with \( r = 1/m, \) but \( x \) is not of the form (1) with \( r = 1/(m-1) \) because \( x > I_{1/(m-1)}(1) = 0 \). Besides requiring \( r > 0 \) we do not make any other restrictions on \( r \).

The following theorem is basic for our considerations.

**Theorem 1:** (S. Kakeya, 1914) Let \( (\lambda_n) \) be a sequence of positive real numbers, such that the series

\[
\sum_{n=1}^{\infty} \lambda_n = s
\]

is convergent with sum \( s \) and the inequalities

\[
\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots
\]

are fulfilled.

Then, each number \( x \in [0, s] \) may be written in the form

\[
x = \sum_{n=1}^{\infty} \epsilon_n \lambda_n \quad (\epsilon_n \in \{0, 1\})
\]
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if and only if

$$\lambda_n \leq \lambda_{n+1} + \lambda_{n+2} + \ldots$$  \hspace{1cm} (7)

for all $n \in N$.

The “digits” $\epsilon_n$ of the expansion may be determined recursively by the following algorithm:
If $n \geq 1$ and if the digits $\epsilon_i$ of the expansion of $x$ are already defined for all $i < n$, then we let

$$\epsilon_n = 1 \text{ if } \sum_{i=1}^{n-1} \epsilon_i \lambda_i + \lambda_n < x.$$  \hspace{1cm} (7a)

Otherwise, we set $\epsilon_n = 0$.

Then, each expansion with $x > 0$ is infinite, i.e. there is an infinite set of integers $n$ with $\epsilon_n = 1$.

A proof of Theorem 1 can be found in [1], or in [7, exercise 131] or in [8].

For our purpose it is practical to introduce the following notion (see [6]):

**Definition:** A sequence $(\lambda_n)$ satisfying conditions (4) and (5) of Theorem 1 is said to be interval-filling (relating to $[0,1]$) if every number $x \in [0,1]$ can be written in the form (6).

**2. THE CASE** $0 < r \leq 1$

First we give an example of an application of

**Theorem 1:** Let $\lambda_n = 1/F_n^r$ for all $n \in N$, where $r$ is a fixed number with $0 < r \leq 1$. As we have mentioned above this sequence satisfies condition (4) of Theorem 1. (5) is also valid. For the proof of (7) we note first that $1/F_n < 2/F_{n+1}$ is valid for all $n \in N$. With $0 < r \leq 1$ we get

$$\frac{1}{F_n} < \frac{2}{F_{n+1}} \leq \frac{2^{1/r}}{F_{n+1}} \text{ which yields } \frac{1}{(F_n)^r} < \frac{2}{(F_{n+1})^r}.$$  

From this we obtain by mathematical induction:

$$1/(F_n)^r - 1/(F_{n+k})^r < 1/(F_{n+1})^r + 1/(F_{n+2})^r + \cdots + 1/(F_{n+k})^r$$

for all $k \geq 1$.

Now let $k \rightarrow \infty$. The limits of the two sides in the preceding inequality exist and we obtain

$$\frac{1}{(F_n)^r} \leq \sum_{k=1}^{\infty} \frac{1}{(F_{n+k})^r}.$$  

Condition (7) is thus established. The application of Theorem 1 immediately yields, that each real number $x$ with $0 < x \leq I_r$, where $0 < r \leq 1$, has (at least) one expansion of the form (1). In other words: $(1/F_n^r)_{n=1}^{\infty}$ is interval-filling relating to $[0, I_r]$.

This statement can be extended considerably.
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Theorem 2: For each real number $x$ with $0 < x < 1$ and fixed $r$ with $0 < r < 1$ the set $C_x$, which consists of all different expansions for $x$ of the form (1), is uncountable; it has cardinality $c$ (the power of the continuum).

The proof is based on an idea which is used in [2] and [3] considering the representation of the real number $x$ in the form

$$x = \sum_{n=1}^{\infty} \epsilon_n q^{-n}$$

with non-integral base $q$. Such an expansion is not unique in general.

Our central point is the construction of a subsequence of $(1/(F_n)^r)_{n=1}^\infty$ which also satisfies the conditions of Theorem 1.

Before we give a proof of Theorem 2 we need some results on sums of Fibonacci reciprocals.

Theorem 3: (Jensen’s inequality, see [5]). Let $0 < r < 1$ and let $A$ be a finite or infinite subset of $N$. Then, we claim that

$$\sum_{i \in A} 1/F_i \leq \left( \sum_{i \in A} 1/(F_i)^r \right)^{1/r}.$$

Proof: Let us let $a = (\sum_{i \in A} 1/(F_i)^r)^{1/r}$. Thus, $\sum_{i \in A} 1/(F_i)^r = 1$ and we get $1/(F_i a)^r = 1$ for all $i \in A$. $1 \geq r$ yields $1/(F_i a)^r \leq 1/(F_i)^r$ for $i \in A$. Therefore,

$$\sum_{i \in A} 1/(F_i a)^r \leq \sum_{i \in A} 1/(F_i)^r = 1.$$

Multiply by $a$. From the definition of $a$ and because of the last inequality we obtain the assertion. $\square$

Theorem 4: Let $0 < r < 1$. Let $z$ denote a positive integer.

(i) If $z = 2k + 1$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+2})^r}.$$ 

(ii) If $z = 2k$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+2})^r} + \frac{1}{(F_{z+3})^r}.$$ 

(iii) If $z = 2k + 1$, then

$$\frac{1}{(F_z)^r} < \frac{1}{(F_{z+2})^r} + \frac{1}{(F_{z+3})^r} + \cdots + \frac{1}{(F_{z+n(z)})^r},$$

with an integer $n(z)$ dependent on the odd integer $z$, with $n(z) \leq n(z+2)$ and $n(2k+1) \to \infty$, as $k \to \infty$.  

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(iv) If \( z = 2k \), then

\[
\frac{1}{(F_z)^r} < \frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+3})^r} + \frac{1}{(F_{z+4})^r} + \cdots + \frac{1}{(F_{z+k})^r}
\]

with \( k = 7 \) if \( z = 2 \) and \( k = 5 \) if \( z \geq 4 \).

**Proof:** First we treat the case \( r = 1 \).

(i) \( z = 2k + 1 \). The assertion is equivalent to \( F_{z+1}F_{z+2} < F_z(F_{z+1} + F_{z+2}) \) or \( F_{z+1}(F_{z+2} - F_z) < F_zF_{z+2} \) or \((F_{z+1})^2 < F_zF_{z+2} \). Then, the well-known formula \( F_n^2 - F_{n+1}F_{n-1} = (-1)^{n+1} \) with \( n = z + 1 \) yields \((F_{z+1})^2 = F_{z+2}F_z - 1 < F_{z+2}F_z \).

The proof of (i) for \( r = 1 \) is complete.

(ii) \( z = 2k \). Using (i), we get

\[
\frac{1}{F_{z+1}} < \frac{1}{F_{z+2}} + \frac{1}{F_{z+3}}
\]

and then

\[
\frac{1}{F_z} < \frac{2}{F_{z+1}} < \frac{1}{F_{z+1}} + \frac{1}{F_{z+2}} + \frac{1}{F_{z+3}}.
\]

(iii) \( z = 2k + 1 \). For the purpose of abbreviation let \( \delta = \beta/\alpha = (\sqrt{5} - 3)/2 \). Then, \( |\delta| < 1 \).

Using the Binet’s formula we have

\[
\frac{F_z}{F_{z+2}} + \frac{F_z}{F_{z+3}} + \cdots + \frac{F_z}{F_{z+n}} = \frac{\alpha^z - \beta^z}{\alpha^{z+2} - \beta^{z+2}} + \cdots + \frac{\alpha^z - \beta^z}{\alpha^{z+n} - \beta^{z+n}}
\]

\[
= \alpha^{-2} \frac{(1 + |\delta|^z)}{1 + |\delta|^z} + \cdots + \alpha^{-n} \frac{(1 + |\delta|^z)}{1 + |\delta|^z} + \cdots + \frac{\alpha^z - \beta^z}{\alpha^{z+n} - \beta^{z+n}}
\]

\[
> \frac{(1 + |\delta|^z)(1 - (\alpha)^{n-1})}{(1 + |\delta|^z)(1 - (\alpha))} \quad \text{(Note } \alpha^2(1 - (1/\alpha)) = 1! \text{)}
\]

\[
= \frac{(1 + |\delta|^z)(1 - (1/\alpha)^{n-1})}{(1 + |\delta|^z)(1 - (1/\alpha))}.
\]

Because \( |\delta| < 1 \) it follows that \((1 + |\delta|^z)/(1 + |\delta|^z) > 1 \).

Further, we notice that the increasing sequence \((1 - (1/\alpha)^{n-1}) \) has limit 1, as \( n \to \infty \). Therefore, it follows that the inequality

\[
\frac{(1 + |\delta|^z)(1 - (1/\alpha)^{n-1})}{(1 + |\delta|^z(1 - (1/\alpha)) > 1}
\]
is valid for all sufficient large values of \( n \in \mathbb{N} \). We denote the minimum of these values by \( n(z) \). Thus, we have

\[
\frac{F_z}{F_{z+2}} + \frac{F_z}{F_{z+3}} + \cdots + \frac{F_z}{F_{z+n}} > 1
\]

for all \( n \geq n(z) \). This is equivalent to (iii).

The assertions \( n(z) \leq n(z + 2) \) and \( n(z + 2k + 1) \to \infty \) as \( k \to \infty \) are easily proved.

(iv) Let \( z = 2k \). If \( z = 2 \) a direct computation leads to the assertion. We observe that for \( z \geq 4 \), the desired result is equivalent to

\[
\frac{F_z F_{z+1}}{F_{z-1}} \left( \frac{1}{F_{z+3}} + \frac{1}{F_{z+4}} + \frac{1}{F_{z+5}} \right) > 1.
\]

Applying (i) with the odd integer \( z + 3 \) to the parenthesis on the left hand side, we obtain

\[
\frac{F_z F_{z+1}}{F_{z-1}} \left( \frac{1}{F_{z+3}} + \frac{1}{F_{z+4}} + \frac{1}{F_{z+5}} \right) > \frac{2F_z F_{z+1}}{F_{z-1} F_{z+3}}.
\]

Therefore, it is enough to establish that \( 2F_z F_{z+1} > F_{z-1} F_{z+3} \). For that purpose we begin with the well-known equation \( F_{n+2} F_{n-1} - F_n F_{n+1} = (-1)^n \) \( (n \in \mathbb{N}) \). We obtain \( F_{z-2} F_{z+1} - F_{z-1} F_z = -1 \) and from this \( 2F_{z+1} F_{z-2} > F_{z-1} F_z \). It follows step-by-step that \( 2F_{z+1}(F_z - F_{z-1}) > F_{z-1} F_z \); \( 2F_z F_{z+1} > 2F_{z-1} F_{z+1} + F_{z-1} F_z = F_{z-1} F_{z+3} \). We have therefore proved all parts of the theorem for \( r = 1 \).

The general assertions for \( 0 < r \leq 1 \) are immediate consequences of Theorem 3. For instance: In the event of (i) the subset \( A \) is as follows: \( A = \{z + 1, z + 2\} \). Then, we have by Theorem 3

\[
\frac{1}{F_z} < \frac{1}{F_{z+1}} + \frac{1}{F_{z+2}} \leq \left( \frac{1}{(F_{z+1})^r} + \frac{1}{(F_{z+2})^r} \right)^{1/r}.
\]

Raising both sides to the \( r^{th} \) power we have (i).

All other cases follow in a similar way. Therefore, the proof of Theorem 4 is complete.

Before we continue with the proof of Theorem 2 let us give a simple application of Theorem 4.

\( r \) will be chosen with \( 0 < r \leq 1 \). Assume that we have a representation of \( x \in [0, \ell_r] \) in the form (1) with the interval-filling sequence \( (1/(F_n)^r)_{n=1}^{\infty} \) on the basis of the algorithm (7a).

**Theorem 5:** Consider the sequence \( (\epsilon_n(x))_{n=1}^{\infty} \) of digits. A chain of consecutive digits “1” following a digit “0” has at most length two.

**Proof:** Let \( \epsilon_n(x) = 0, \epsilon_{n+1}(x) = 1, \epsilon_{n+2}(x) = 1, \ldots, \epsilon_{n+k}(x) = 1 \) be a chain of the described kind. Then, we obtain by algorithm (7a)

\[
\sum_{i=1}^{n-1} \frac{\epsilon_i}{(F_i)^r} + \frac{1}{(F_{n+1})^r} + \cdots + \frac{1}{(F_{n+k})^r} < x \leq \sum_{i=1}^{n-1} \frac{\epsilon_i}{(F_i)^r} + \frac{1}{(F_n)^r}.
\]
Thus,

\[
\frac{1}{(F_n)^r} > \frac{1}{(F_{n+1})^r} + \cdots + \frac{1}{(F_{n+k})^r}.
\]

We now appeal to Theorem 4. It implies that \( k \) must be equal to 1 (at most equal to 2), if \( n \) is an odd (even) number since the assumption \( k \geq 2 \) (\( k \geq 3 \)) leads to a contradiction with Theorem 4(i) or (ii).

The proof is complete. \( \square \)

**Proof of Theorem 2:** We choose a sequence of even integers \( (z_j)_{j=1}^\infty = (2k_j)_{j=1}^\infty \) with \( z_{j+1} - z_j > \max\{9, n(z_j - 1)\} \) for all \( j \in N \). The first member \( z_1 \) will be chosen (later) to be sufficiently large. Let \( M = N - \{z_j\}_{j=1}^\infty \). Consider the set \( \{1/(F_m)^r : m \in M\} \) as a non-increasing sequence \( (\lambda_n)_{n=1}^\infty \) of numbers: \( \lambda_1 = 1/(F_1)^r, \lambda_2 = 1/(F_2)^r, \ldots, \lambda_{z_1 - 1} = 1/(F_{z_1 - 1})^r, \lambda_{z_1} = 1/(F_{z_1 + 1})^r, \lambda_{z_1 + 1} = 1/(F_{z_1 + 2})^r, \ldots \).

Next, we shall show that Theorem 1 is applicable to the sequence \( (\lambda_n)_{n=1}^\infty \), in particular the validity of (6).

First we determine for each \( m \in M \) the unique number \( j \in N \) such that the condition \( z_{j-1} + 1 \leq m \leq z_j - 1 \) is satisfied \( (z_0 = 0) \). Then, we obtain with the help of Theorem 4

\[
\sum_{n > m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+1})^r} + \frac{1}{(F_{m+2})^r} + \frac{1}{(F_{m+3})^r} > \frac{1}{(F_m)^r},
\]

\( n \in M \) if \( z_{j-1} + 1 \leq m \leq z_j - 4 \) in view of Theorem 4(i), (ii);

\[
\sum_{n > m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+1})^r} + \frac{1}{(F_{m+2})^r} + \frac{1}{(F_{m+3})^r} > \frac{1}{(F_m)^r},
\]

\( n \in M \) if \( m = z_j - 3 \) in view of Theorem 4(i);

\[
\sum_{n > m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+1})^r} + \frac{1}{(F_{m+2})^r} + \cdots + \frac{1}{(F_{m+k})^r} > \frac{1}{(F_m)^r},
\]

\( n \in M \) if \( m = z_j - 2 \) in view of Theorem 4(iv); and

\[
\sum_{n > m} \frac{1}{(F_n)^r} > \frac{1}{(F_{m+1})^r} + \frac{1}{(F_{m+2})^r} + \cdots + \frac{1}{(F_{m+n(m)})^r} > \frac{1}{(F_m)^r},
\]

\( n \in M \) if \( m = z_j - 1 \) in view of Theorem 4(iii).

So, we obtain for each \( m \in M : 1/(F_m)^r < \sum_{n > m, n \in M} 1/(F_n)^r \), that is we proved that condition (7) of Theorem 1 is satisfied. It is clear that (4) and (5) are valid.

Let \( 0 < x < I_r \). We choose \( z_1 \) so that the following conditions are satisfied simultaneously:

\[
(*) \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r} < x \text{ and } x + \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r} < I_r.
\]
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This is possible, because \( \lim_{z_1 \to \infty} \sum_{n \geq z_1} 1/(F_n)^r = 0 \). Let \( \Delta \) be any subset of the set \( \{ z_j \}_{j=1}^{\infty} \). We define now the 0-1-sequence \( (\delta_j)_{j=1}^{\infty} \) in the following way: \( \delta_j = 1 \), if \( z_j \in \Delta \), \( \delta_j = 0 \), if \( z_j \notin \Delta \). Consider the number

\[
y = x - \sum_{j=1}^{\infty} \frac{\delta_j}{(F_{z_j})^r}.
\]

We obtain from the above conditions (*) that

\[
y \geq x - \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r} > 0 \quad \text{and} \quad y \leq x < I_r - \sum_{j=1}^{\infty} \frac{1}{(F_{z_j})^r}.
\]

It follows from this that \( 0 < y < \sum_{n=1}^{\infty} \lambda_n \).

Now the key point is the application of Theorem 1. For each real number in the interval \( [0, \sum_{n=1}^{\infty} \lambda_n] \) there is a series of the form

\[
\sum_{n=1}^{\infty} \epsilon_n \lambda_n \quad \epsilon_n \in \{0, 1\}.
\]

With a view to the definition of \( y \) we receive the following representation:

\[
x = \sum_{n=1}^{\infty} \epsilon_n \lambda_n + \sum_{j=1}^{\infty} \epsilon_j \delta_j.
\]

We note that a Fibonacci reciprocal contained in the second sum cannot occur in the first, which implies that the representation of \( x \) is dependent on the sequence \( (\delta_j) \). Two different sequence \( (\delta_j) \) and \( (\delta'_j) \) lead to different representations of \( x \). It is well-known that the set of all 0-1-sequences has cardinality \( c \) (the power of the continuum). Therefore, the set \( C_x \) of different representations of \( x \) in the form (1) has at least cardinality \( c \). Because the cardinality of the set of 0-1-sequences equals the cardinality of the continuum, the set \( C_x \) has cardinality at most \( c \).

Theorem 2 is thus established. \( \square \)

Next, we will draw a comparison between our Theorem 2 and results in [2] and [3], which are due to P. Erdős, M. Horvath, I. Joó and V. Komornik.

First we make the observation that by Binet's formula \( \lim_{n \to \infty} F_n/\alpha^n = 1/\sqrt{5} \), that is \( F_n \) and \( \alpha^n \) are "almost" proportional as \( n \to \infty \). To simplify matters we assume \( F_n \sim \alpha^n \). Then it follows that \( (F_n)^r \sim \alpha^{nr} = q^n \) with \( q = \alpha^r \) and the interval \( 0 < r \leq 1 \) corresponds to the interval \( 1 < q \leq \alpha \).

Let \( q \in (1, \alpha) \). It was proven in [2] (see Theorem 3 in [2]) that for every \( x \in (0, 1/q - 1) \) there are \( c \) different expansions of the form

\[
x = \sum_{n=1}^{\infty} \epsilon_n q^n \quad \epsilon_n \in \{0, 1\}.
\]
We can say that this result is analogous to our Theorem 2, if we take into consideration the above-mentioned remark on \((F_n)^r\) and \(q^r\).

On the other hand it was shown in [3] (see the proof of Theorem 1 in [3]) that, if we assume in (8) \(x = 1\) and \(q = \alpha\), there exist precisely countably many expansions of the form (8). It is surprising that we have different cardinal numbers relating the set of representations for \(x = 1\) and \(r = 1\) according to (1) and the set of representation for \(x = 1\) and \(q = \alpha\) according to (8).

3. THE CASE \(r > 1\)

We shall prove two further theorems regarding expansions of the form (1).

**Theorem 6:** Let \(r\) satisfy \(1 < r < \log 2/\log \alpha\). Then, there is an even integer \(m(r)\) such that the sequence \((1/(F_n)^r)_{n=m(r)-1}^\infty\) is interval-filling.

**Theorem 7:** Let \(r\) satisfy \(r \geq \log 2/\log \alpha\). Then, there is no integer \(m \in N\) such that \((1/(F_n)^r)_{n=m}^\infty\) is an interval-filling sequence.

**Proof of Theorem 6:** In view of the equation \(\beta/\alpha = -1/\alpha^2\) and with the help of Binet’s formula it easily follows that

\[
F_{n+1}/F_n = \alpha E(n) \text{ where } E(n) = \frac{1 + (-1)^{n+2}\alpha^{-2n-2}}{1 + (-1)^{n+1}\alpha^{-2n}}.
\]

If \(1 < r < \log 2/\log \alpha\) holds, then \(2 > 2^{1/r} > \alpha\). As soon as \(n\) is an odd integer we get \(E(n) < 1\). Thus, it follows that \(F_{n+1}/F_n < 2^{1/r}\) for an odd integer \(n\). On the other hand, we obtain from the definition of \(E(n)\) that for even integers the following statements are valid: \(E(n) > 1, E(n) > E(n + 2), \lim_{n \to \infty} E(n) = 1\). Hence, there is a smallest even integer \(m(r)\) such that \(1 < E(m(r)) < 2^{1/r}/\alpha\). Therefore, \(F_{n+1}/F_n = \alpha E(n) \leq \alpha E(m(r)) < 2^{1/r}\) for each even \(n \geq m(r)\). Summarizing we obtain \(F_{n+1}/F_n < 2^{1/r}\) or \((F_{n+1})^r/(F_n)^r < 2\) for all integers \(n \geq m(r) - 1\). This implies that Theorem 1 is applicable since the sequence \((1/(F_n)^r)\) with \(n \geq m(r) - 1\) meets all the requirements of the theorem, in particular condition (7). Theorem 6 is thus established.

**Proof of Theorem 7:** Let \(r \geq \log 2/\log \alpha\), equivalent to \(\alpha^r \geq 2\). First we shall prove that, for all even integers \(n \in N\), we have

\[
1/(F_n)^r > 1/(F_{n+1})^r + 2/(F_{n+2})^r.
\]

We again use the definition of \(E(n)\) in the proof of Theorem 6. We receive from (10) the equivalent inequality

\[
\alpha^r (E(n)E(n+1))^r > (E(n+1))^r + 2/\alpha^r \quad (n \text{ even}).
\]

On the other side, we obtain for even \(n \in N:\)

\[
E(n)E(n+1) = 1 + \frac{1 - 1/\alpha^4}{\alpha^{2n} - 1} > 1 \text{ and } E(n+1) < 1.
\]
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The last two inequalities and \( \alpha^r \geq 2 \) yield that (11) is valid for all even \( n \in \mathbb{N} \), because

\[
\alpha^r (E(n)E(n+1))^r \geq 2 ((E(n)E(n+1))^r > 2 > (E(n+1))^r + 1 \geq (E(n+1))^r + 2/\alpha^r.
\]

Thus, the equivalent statement (10) follows, from which we obtain by mathematical induction:

\[
\frac{1}{(F_n)^r} - \frac{1}{(F_{n+1})^r} > \sum_{i=1}^{2k} \frac{1}{(F_{n+i})^r}
\]

(12)

for all \( k \geq 1 \) and even \( n \).

Then, it follows from (12) as \( k \to \infty \):

\[
\frac{1}{(F_n)^r} \geq \sum_{i=1}^{\infty} \frac{1}{(F_{n+i})^r} \quad (n \in \mathbb{N}, \text{ } n \text{ even}).
\]

(13)

Now, suppose that in (13) for two consecutive even numbers \( n = v \) and \( n = v + 2 \) the equals sign is valid.

Then, a simple calculation shows that we have a contradiction to (10):

\[
\frac{1}{(F_v)^r} = \frac{1}{(F_{v+1})^r} + \frac{2}{(F_{v+2})^r},
\]

i.e. from two successive inequalities (13) there is at most one equality. Next, consider the set

\[
A(r) = \{ n | n \in 2\mathbb{N}, \ 1/(F_n)^r > \sum_{i=1}^{\infty} 1/(F_{n+i})^r \}.
\]

From the preceding argument it is clear that \( A(r) \) is an infinite subset of \( \mathbb{N} \), such that condition (7) of Theorem 1 is not true for \( n \in A(r) \). We conclude that there is no integer \( m \in \mathbb{N} \), such that the sequence \( 1/(F_n)^r \) is interval-filling. \( \square \)

REFERENCES

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AMS Classification Numbers: 03E17, 11A17, 11B39

Please Send in Proposals!

The Eleventh International Conference on Fibonacci Numbers and their Applications

July 5 – July 9, 2004
Technical University Carolo-Wilhelmina,
Braunschweig, Germany

Local Organizer: H. Harborth
Conference Organizer: W. Webb

Call for Papers: The purpose of the conference is to bring together people from all branches of mathematics and science with interests in recurrence sequences, their applications and generalizations, and other special number sequences.

Deadline: Papers and abstracts should be submitted in duplicate to W. Webb by May 1, 2004 at:

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Pullman, WA 99164-3113
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Phone: 509-335-3150

Electronic submissions, preferably in AMS – TeX, sent to webb@math.wsu.edu

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International Committee: A. Adelberg (U.S.A.), M. Bicknell-Johnson (U.S.A.), C. Cooper (U.S.A.), Y. Horibe (Japan), A. Horadam (co-chair) (Australia), J. Lahr (Luxembourg), A. Philippou (co-chair) (Greece), G. Phillips (co-chair) (Scotland), A. Shannon (Australia), L. Somer (U.S.A.), J. Turner (New Zealand).

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Information: www.mscs.dal.ca/fibonacci/eleventh.html
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