HEPTAGONAL NUMBERS IN THE FIBONACCI SEQUENCE
AND
DIOPHANTINE EQUATIONS

4x^2 = 5y^2(5y - 3)^2 \pm 16

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1. INTRODUCTION

The numbers of the form \( \frac{m(5m-3)}{2} \), where \( m \) is any positive integer, are called heptagonal numbers. The first few are 1, 7, 18, 34, 55, 81, …, and are listed in [4] as sequence number A000566. In this paper it is established that 0, 1, 13, 34 and 55 are the only generalized heptagonal numbers (where \( m \) is any integer) in the Fibonacci sequence \( \{F_n\} \). These numbers can also solve the Diophantine equations of the title. Earlier, J.H.E. Cohn [1] has identified the squares and Ming Luo (see [2] and [3]) has identified the triangular, pentagonal numbers in \( \{F_n\} \). Furthermore, in [5] it is proved that 1, 4, 7 and 18 are the only generalized heptagonal numbers in the Lucas sequence \( \{L_n\} \).

2. IDENTITIES AND PRELIMINARY LEMMAS

We have the following well known properties of \( \{F_n\} \) and \( \{L_n\} \):

\[
F_{-n} = (-1)^{n+1}F_n \quad \text{and} \quad L_{-n} = (-1)^n L_n \tag{1}
\]

\[
2F_{m+n} = F_m L_n + F_n L_m \quad \text{and} \quad 2L_{m+n} = 5F_m F_n + L_m L_n \tag{2}
\]

\[
F_{2n} = F_n L_n \quad \text{and} \quad L_{2n} = L_n^2 + 2(-1)^{n+1} \tag{3}
\]

\[
L_n^2 = 5F_n^2 + 4(-1)^n \tag{4}
\]

\[
2|F_n \iff 3|n \quad \text{and} \quad 2|L_n \iff 3|n \tag{5}
\]

\[
3|F_n \iff 4|n \quad \text{and} \quad 3|L_n \iff n \equiv 2(\mod 4) \tag{6}
\]

\[
9|F_n \iff 12|n \quad \text{and} \quad 9|L_n \iff n \equiv 6(\mod 12) \tag{7}
\]

\[
L_{8n} \equiv 2(\mod 3). \tag{8}
\]

If \( m \equiv \pm 2(\mod 6) \), then

\[
F_m \equiv 3(\mod 4) \quad \text{and} \quad L_{2m} \equiv 7(\mod 8), \tag{9}
\]

\[
F_{2mt+n} \equiv (-1)^t F_n (\mod L_m), \tag{10}
\]

where \( n, m, \) and \( t \) denote integers.

Since, \( N \) is a generalized heptagonal number if and only if \( 40N + 9 \) is the square of an integer congruent to \( 7(\mod 10) \), we identify those \( n \) for which \( 40F_n + 9 \) is a perfect square. We begin with

Lemma 1: Suppose \( n \equiv 0(\mod 2^4 \cdot 17) \). Then \( 40F_n + 9 \) is a perfect square if and only if \( n = 0 \).

Proof: If \( n = 0 \), then \( 40F_n + 9 = 3^2 \).
Conversely, suppose \( n \equiv 0 \pmod{2^4 \cdot 17} \) and \( n \neq 0 \). Then \( n \) can be written as \( n = 2 \cdot 17 \cdot 2^\theta \cdot g \), where \( \theta \geq 3 \) and \( 2 \nmid g \). And since for \( \theta \geq 3, 2^\theta + 8 \equiv 2^\theta \pmod{680} \), taking \( k = 2^\theta \) if \( \theta \equiv 0, 5 \) or \( 7 \pmod{8} \) and \( k = 17 \cdot 2^\theta \) for the other values of \( \theta \), we have

\[
k \equiv 32, 128, \pm 136, 256, 272 \text{ or } 408 \pmod{680}.
\]

Since \( k \equiv \pm 2 \pmod{6} \), from (10), we get

\[
40F_n + 9 = 40F_{2k(2x+1)} + 9 \equiv 40(-1)^x F_{2k} + 9 \pmod{L_{2k}}.
\]

Therefore, using properties (1) to (9) of \( \{F_n\} \) and \( \{L_n\} \), the Jacobi symbol

\[
\left( \frac{40F_n + 9}{L_{2k}} \right) = \left( \frac{\pm 40F_{2k} + 9}{L_{2k}} \right) = \left( \frac{3}{L_{2k}} \right) \left( \frac{\pm 40F_{2k} + 3}{L_{2k}} \right) = - \left( \frac{L_{2k}}{3} \right) \left( \frac{\pm 80F_k L_k + 3L_k^2}{L_{2k}} \right).
\]

Letting \( u_k = \frac{F_k}{3} \) and \( v_k = 80u_k \pm 3L_k \) we obtain

\[
\left( \frac{40F_n + 9}{L_{2k}} \right) = \pm \left( \frac{80u_k L_k \pm 3L_k^2}{80u_k L_k \pm 3L_k^2} \right) = - \left( \frac{L_{2k}}{80u_k L_k \pm 3L_k^2} \right) = - \left( \frac{L_{2k}}{L_k} \right) \left( \frac{L_{2k}}{v_k} \right)
\]

\[
= - \left( \frac{-2}{L_k} \right) \left( \frac{1}{2} F_k^2 + L_k^2 \right) = \left( \frac{2}{L_k \cdot v_k} \right) \left( \frac{720F_k^2 + 144L_k^2}{v_k} \right)
\]

Since \( v_k = \frac{80F_k}{3} \pm 3L_k \), then \( 144L_k^2 \equiv \frac{102400F_k^2}{9} \pmod{v_k} \) and

\[
\left( \frac{720F_k^2 + 144L_k^2}{v_k} \right) = \left( \frac{108800F_k^2}{v_k} \right) = \left( \frac{5 \times 1361}{v_k} \right) = \left( \frac{5}{v_k} \right) \left( \frac{v_k}{1361} \right) = \left( \frac{v_k}{1361} \right)
\]

\[
= - \left( \frac{80F_k \pm 9L_k}{1361} \right).
\]

Furthermore, \( \left( \frac{2}{L_k \cdot v_k} \right) = -1 \), it follows that \( \left( \frac{40F_n + 9}{L_{2k}} \right) = \left( \frac{80F_k \pm 9L_k}{1361} \right) \).

But modulo 1361, the sequence \( \{80F_n \pm 9L_n\} \) is periodic with period 680 and by (11), \( \left( \frac{80F_k \pm 9L_k}{1361} \right) = -1, \) for all values of \( k \). The lemma follows.

**Lemma 2:** Suppose \( n \equiv \pm 1, \ 2, \ \pm 7, \ \pm 9, \ 10 \pmod{133280} \). Then \( 40F_n + 9 \) is a perfect square if and only if \( n = \pm 1, \ 2, \ \pm 7, \ \pm 9, \ 10 \).

**Proof:** To prove this, we adopt the following procedure which enables us to tabulate the corresponding values reducing repetition and space.

Suppose \( n \equiv \varepsilon \pmod{N} \) and \( n \neq \varepsilon \). Then \( n \) can be written as \( n = 2 \cdot \delta \cdot 2^\theta \cdot g + \varepsilon \), where \( \theta \geq 7 \) and \( 2 \nmid g \). Then, \( n = 2km + \varepsilon \), where \( k \) is odd, and \( m \) is even.

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Now, using (10), we choose $m$ such that $m \equiv \pm 2 \pmod{6}$. Thus,

$$40F_n + 9 = 40F_{2m+e} + 9 \equiv 40(-1)^k F_e + 9 \pmod{I_m}.$$ 

Therefore, the Jacobi symbol

$$\left(\frac{40F_n + 9}{L_m}\right) = \left(\frac{-40F_e + 9}{L_m}\right) = \left(\frac{I_m}{M}\right). \quad (12)$$

But modulo $M$, $\{L_n\}$ is periodic with period $P$. Now, since for $\theta \geq \gamma, 2^{\theta+s} \equiv 2^\theta \pmod{P}$, choosing $m = \mu \cdot 2^\theta$ if $\theta \equiv \zeta(\pmod{s})$ and $m = 2^\theta$ otherwise, we have $m \equiv c(\pmod{P})$ and $(\frac{L_m}{M}) = -1$, for all values of $m$. From (12), it follows that $(\frac{40F_n + 9}{L_m}) = -1$, for $n \neq \varepsilon$. For each value of $\varepsilon$, the corresponding values are tabulated in this way (Table A).

<table>
<thead>
<tr>
<th>$\varepsilon$</th>
<th>$N$</th>
<th>$\delta$</th>
<th>$\gamma$</th>
<th>$s$</th>
<th>$M$</th>
<th>$P$</th>
<th>$\mu$</th>
<th>$\zeta(\pmod{s})$</th>
<th>$c(\pmod{P})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pm 1, \pm 2$</td>
<td>$2^2 \cdot 7^2$</td>
<td>$7^2$</td>
<td>1</td>
<td>4</td>
<td>31</td>
<td>30</td>
<td>$7^2$</td>
<td>2, 3.</td>
<td>2, 16.</td>
</tr>
<tr>
<td>$\pm 7$</td>
<td>$2^5 \cdot 7^2$</td>
<td>$7^2$</td>
<td>4</td>
<td>36</td>
<td>511</td>
<td>592</td>
<td>$7^2$</td>
<td>0, 1, 6, 7, 8,</td>
<td>$\pm 16, \pm 32, \pm 48, \pm 64, \pm 112, \pm 160, \pm 192, \pm 208, \pm 240, \pm 272, \pm 288$.</td>
</tr>
<tr>
<td>$\pm 9$</td>
<td>$2^5 \cdot 5 \cdot 7^2$</td>
<td>$5^2 \cdot 7^2$</td>
<td>4</td>
<td>48</td>
<td>1351</td>
<td>1552</td>
<td>$5^2 \cdot 7^2$</td>
<td>2, 20, 26, 44.</td>
<td>$\pm 32, \pm 48, \pm 64, \pm 112, \pm 128, \pm 192, \pm 256, \pm 304, \pm 352, \pm 368, \pm 432, \pm 464, \pm 480, \pm 528, \pm 560, \pm 592, \pm 672, \pm 688, \pm 704, \pm 752, \pm 768$.</td>
</tr>
<tr>
<td>$10$</td>
<td>$2^2 \cdot 7^2 \cdot 17$</td>
<td>$17^2 \cdot 7^2$</td>
<td>4</td>
<td>52</td>
<td>2191</td>
<td>2512</td>
<td>$17^2$</td>
<td>0, 8, 26, 34.</td>
<td>$\pm 32, \pm 48, \pm 112, \pm 128, \pm 192, \pm 224, \pm 272, \pm 432, \pm 448, \pm 512, \pm 624, \pm 1024, \pm 1040, \pm 1072, \pm 1248$.</td>
</tr>
</tbody>
</table>
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Since L.C.M. of \((2^2 \cdot 7^2, 2^5 \cdot 7^2, 2^5 \cdot 5 \cdot 7^2, 2^5 \cdot 7^2 \cdot 17) = 133280\), the lemma follows.

As a consequence of Lemma 1 and 2 we have the following.

**Corollary 1:** Suppose \(n \equiv 0, \pm 1, 2, \pm 7, \pm 9, 10 (\mod 133280)\). Then \(40F_n + 9\) is a perfect square if and only if \(n = 0, \pm 1, 2, \pm 7, \pm 9, 10\).

**Lemma 3:** \(40F_n + 9\) is not a perfect square if \(n \not\equiv 0, \pm 1, 2, \pm 7, \pm 9, 10 (\mod 133280)\).

**Proof:** We prove the lemma in different steps eliminating at each stage certain integers \(n\) congruent modulo 133280 for which \(40F_n + 9\) is not a square. In each step, we choose an integer \(m\) such that the period \(p\) (of the sequence \(\{F_n\} \mod m\)) is a divisor of 133280 and thereby eliminate certain residue class modulo \(p\). For example

**Mod 29:** The sequence \(\{F_n\} \mod 29\) has period 14. We can eliminate \(n \equiv \pm 3, \pm 6\) and 12 (mod 14), since \(40F_n + 9 \equiv 2, 10, 8\) and \(27(\mod 29)\) respectively and they are quadratic nonresidue modulo 29. There remain \(n \equiv 0, \pm 1, 2, \pm 4, \pm 5, \pm 7, \pm 9, \pm 10, \pm 13, 14\) or \(16(\mod 28)\).

Similarly we can eliminate the remaining values of \(n\). After reaching modulus 133280, if there remain any values of \(n\) we eliminate them in the higher modulus (that is in the multiples of 133280). We tabulate them in the following way (Table B).
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<table>
<thead>
<tr>
<th>Period $p$</th>
<th>Modulus $m$</th>
<th>Required values of $n$ where $\frac{40F_n^2 + 9}{m} = -1$</th>
<th>Left out values of $n$ (mod $k$) where $k$ is a positive integer</th>
</tr>
</thead>
<tbody>
<tr>
<td>14</td>
<td>29</td>
<td>$\pm 3, \pm 6, 12.$</td>
<td>$0, \pm 1, 2, \pm 4, \pm 5, 7$ (mod 14)</td>
</tr>
<tr>
<td>28</td>
<td>13</td>
<td>$\pm 13, 16, 18, 24.$</td>
<td>$0, \pm 1, 2, 4, \pm 5, \pm 7, \pm 9, 10, 14$ (mod 28)</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>$\pm 3, 6.$</td>
<td>$0, \pm 1, 2, \pm 7, \pm 9, 10, \pm 23, 28,$</td>
</tr>
<tr>
<td>56</td>
<td>281</td>
<td>4, 42.</td>
<td>$32$ (mod 56)</td>
</tr>
<tr>
<td>16</td>
<td>7</td>
<td>4.</td>
<td>$0, \pm 1, 2, \pm 7, \pm 9, 10, \pm 23, 28,$</td>
</tr>
<tr>
<td>112</td>
<td>14503</td>
<td>$32, \pm 47, \pm 49, \pm 55, 58, 66, 88.$</td>
<td>$\pm 33, 56$ (mod 112)</td>
</tr>
<tr>
<td>32</td>
<td>47</td>
<td>12, 24, 28.</td>
<td>$0, \pm 1, 2, \pm 7, \pm 9, 10, \pm 23, \pm 33,$</td>
</tr>
<tr>
<td>10</td>
<td>11</td>
<td>$\pm 4, 8.$</td>
<td>$\pm 79, \pm 89, \pm 103, \pm 105, \pm 111,$ $112, 114, 168$ (mod 224)</td>
</tr>
<tr>
<td>40</td>
<td>41</td>
<td>$\pm 15, \pm 17, 32.$</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>71</td>
<td>$\pm 19, \pm 21, \pm 23, \pm 27, \pm 33.$</td>
<td>$0, \pm 1, 2, \pm 7, \pm 9, 10, \pm 551,$ $560, 1010$ (mod 1120)</td>
</tr>
<tr>
<td>160</td>
<td>1601</td>
<td>$\pm 39, 40, 90, 122, 130.$</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>2161</td>
<td>$\pm 41, 42.$</td>
<td></td>
</tr>
<tr>
<td>140</td>
<td>141961</td>
<td>$\pm 61.$</td>
<td></td>
</tr>
<tr>
<td>196</td>
<td>97</td>
<td>$\pm 19, \pm 27, 28, \pm 29, \pm 35, 56, \pm 57, \pm 65, 66,$ $86, \pm 91, 122, 150, 178.$</td>
<td>$0, \pm 1, 2, \pm 7, \pm 9, 10, \pm 3369,$ $3911, 3920$ (mod 7840)</td>
</tr>
<tr>
<td>490</td>
<td>491</td>
<td>$72, \pm 77, 100, \pm 133, \pm 141, 142, \pm 147,$ $170, \pm 201, \pm 209, 210, 212, \pm 219, 310,$ $352, 430.$</td>
<td>$0, \pm 1, 2, \pm 7, \pm 9, 10, \pm 3369,$ $3911, 3920$ (mod 7840)</td>
</tr>
<tr>
<td>392</td>
<td>5881</td>
<td>$58, \pm 113, 168.$</td>
<td></td>
</tr>
<tr>
<td>7840</td>
<td>54881</td>
<td>$\pm 551.$</td>
<td></td>
</tr>
<tr>
<td>136</td>
<td>67</td>
<td>$8, \pm 17, \pm 23, \pm 25, 26, 32, 34, \pm 39, 40,$ $\pm 41, 42, 48, \pm 55, \pm 56, \pm 65, 90, 112, 114.$</td>
<td>$0, \pm 1, 2, \pm 7, \pm 9, 10, 66640$ (mod 133280)</td>
</tr>
<tr>
<td>238</td>
<td>239</td>
<td>$\pm 19, 24, 28, \pm 35, \pm 37, \pm 41, \pm 43, 44, \pm 49,$ $\pm 57, \pm 69, 70, \pm 71, \pm 75, \pm 77, 86, 100,$ $\pm 103, \pm 107, 108, 142, 154, 164, 184,$ $196, 206.$</td>
<td>$0, \pm 1, 2, \pm 7, \pm 9, 10, 66640$ (mod 133280)</td>
</tr>
<tr>
<td>680</td>
<td>1361</td>
<td>$\pm 73, \pm 121, \pm 151, \pm 167, \pm 193, \pm 319,$ $\pm 321.$</td>
<td>$0, \pm 1, 2, \pm 7, \pm 9, 10, 66640$ (mod 133280)</td>
</tr>
<tr>
<td>68</td>
<td>1597</td>
<td>$\pm 5, \pm 11, \pm 14, 20, 38, 64.$</td>
<td></td>
</tr>
<tr>
<td>2380</td>
<td>2381</td>
<td>$560, \pm 973, 1962, 2102.$</td>
<td></td>
</tr>
<tr>
<td>34</td>
<td>3571</td>
<td>$\pm 4, \pm 13, 32.$</td>
<td></td>
</tr>
<tr>
<td>1360</td>
<td>5441</td>
<td>$160, 322, 970.$</td>
<td></td>
</tr>
<tr>
<td>8330</td>
<td>16661</td>
<td>$\pm 919, \pm 1461, 7360.$</td>
<td></td>
</tr>
<tr>
<td>124951</td>
<td></td>
<td>$\pm 2389.$</td>
<td></td>
</tr>
<tr>
<td>26656</td>
<td>39983</td>
<td>$\pm 13319.$</td>
<td></td>
</tr>
</tbody>
</table>

Table B

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We now eliminate \( n = 66640 \equiv 133280 \) (mod 133280), equivalently, \( n = 199920 \) (mod 266560). Now, modulo 449, the sequence \( \{40F_n + 9\} \) is periodic with period 448. Also, \( 66640 \equiv 336 \equiv 112 \equiv 448 \) (mod 448), \( (40F_{123} + 9) = -1 \) and \( 199920 \equiv 112 \) (mod 448), \( (40F_{123} + 9) = -1 \). The lemma follows.

3. MAIN THEOREM

Theorem 1: (a) \( F_n \) is a generalized heptagonal number only for \( n = 0, \pm 1, 2, \pm 7, \pm 9 \) or 10; and (b) \( F_n \) is a heptagonal number only for \( n = \pm 1, 2, \pm 9 \) or 10.

Proof: Part (a) of the theorem follows from Corollary 1 and Lemma 3. For part (b), since, an integer \( N \) is heptagonal if and only if \( 40N + 9 = (10m - 3)^2 \) where \( m \) is a positive integer. We have the following table.

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>( \pm 1 )</th>
<th>2</th>
<th>( \pm 7 )</th>
<th>( \pm 9 )</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_n )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>13</td>
<td>34</td>
<td>55</td>
</tr>
<tr>
<td>( 40F_n + 9 )</td>
<td>( 3^2 )</td>
<td>( 7^2 )</td>
<td>( 7^2 )</td>
<td>( 23^2 )</td>
<td>( 37^2 )</td>
<td>( 47^2 )</td>
</tr>
<tr>
<td>( m )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-2</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>( L_n )</td>
<td>2</td>
<td>( \pm 1 )</td>
<td>3</td>
<td>( \pm 29 )</td>
<td>( \pm 76 )</td>
<td>123</td>
</tr>
</tbody>
</table>

Table C.

4. SOLUTIONS OF CERTAIN DIOPHANTINE EQUATIONS

It is well known that if \( x_1 + y_1\sqrt{D} \) (where \( D \) is not a perfect square and \( x_1, y_1 \) are least positive integers) is the fundamental solution of Pell’s equation \( x^2 - Dy^2 = \pm 1 \), then the general solution is given by \( x_n + y_n\sqrt{D} = (x_1 + y_1\sqrt{D})^n \). Therefore, by (4), we have

\[ L_{2n} + \sqrt{5}F_{2n} \text{ is a solution of } x^2 - 5y^2 = 4, \]

or

\[ L_{2n+1} + \sqrt{5}F_{2n+1} \text{ is a solution of } x^2 - 5y^2 = -4. \]

We have, by (13), (14), Theorem 1, and Table C, the following two corollaries.

Corollary 2: The solution set of the Diophantine equation \( 4x^2 = 5y^2(5y - 3)^2 - 16 \) is \( \{(\pm 1, 1), (\pm 29, -2), (\pm 76, 4)\} \).

Corollary 3: The solution set of the Diophantine equation \( 4x^2 = 5y^2(5y - 3)^2 + 16 \) is \( \{(\pm 2, 0), (\pm 3, 1), (\pm 123, 5)\} \).

REFERENCES


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