THE Q MATRIX AS A COUNTEREXAMPLE IN GROUP THEORY

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If g is an element of a group G, then $o(g)$, the order of g, is defined to be the number of distinct elements of G in the set $\{e, g^{\pm 1}, g^{\pm 2}, \cdots\}$, where e is the identity of G. This is equivalent to defining $o(g)$ to be the number of elements in the cyclic subgroup of G generated by g. It is an easy consequence that the order of g equals the least positive integer n such that $g^n = e$. If no such integer exists, g is said to be of infinite order.

In an abelian group H (i.e., $ab = ba$ for all $a, b \in H$) it is easy to show that the product of two elements of finite order must again be of finite order. Indeed, if $o(a) = m$, $o(b) = n$ for some $a, b \in H$, then $(ab)^{mn} = (a^m)^n (b^n)^m = e^n e^m = e$, so $o(ab) \leq mn$. However, this does not necessarily hold in general, as shown in the following counterexample involving the Q matrix.

Let G be the multiplicative group of all nonsingular 2x2 matrices, and let

$$R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

be elements of G. One can check that $R^2 = S^3 = I$, the identity matrix, so that R and S are of finite order. But

$$RS = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = Q ,$$

the Q matrix. Now Basin and Hoggatt [1] have shown that

$$(RS)^n = Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \neq I$$

for any $n > 0$. Thus RS has infinite order.

(See page 80 for reference.)