ITERATED FIBONACCI AND LUCAS SUBSCRIPTS

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Raymond Whitney [3] has proposed the problem of finding recurrence relations for the sequences $U_n = F_{F_n}$, $V_n = F_{L_n}$, $W_n = L_{L_n}$, and $X_n = L_{F_n}$, where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively. In this note we give the required recurrence relations for more general sequences of the form $Y_n = F_{H_n}$, $Z_n = L_{H_n}$, where the H_n are generalized Fibonacci numbers introduced by Horadam.

We will make use of several identities. It follows from the Binet forms for Fibonacci and Lucas numbers that

$$2F_{n+1} = F_n + L_n \quad , \tag{1}$$

$$F_{n-1} = \tfrac{1}{2}(L_n - F_n) \quad , \tag{2}$$

$$L_n^2 - 5F_n^2 = 4(-1)^n \quad , \tag{3}$$

$$2L_{n+1} = 5F_n + L_n \quad . \tag{4}$$

From these H. H. Ferns [1] has shown

$$F_{n+1} = \tfrac{1}{2}\left(\sqrt{5F_n^2 + 4(-1)^n} + F_n\right) \quad , \tag{5}$$

$$L_{n+1} = \tfrac{1}{2}\left(\sqrt{5L_n^2 - 20(-1)^n} + L_n\right) \quad . \tag{6}$$

Equation (5) implies

$$F_{n-1} = \tfrac{1}{2}\left(\sqrt{5F_n^2 + 4(-1)^n} - F_n\right) \quad . \tag{7}$$

We shall also require

$$F_{m+n+1} = F_m F_n + F_{m+1} F_{n+1} \quad , \tag{8}$$

$$L_{m+n+1} = F_m L_n + F_{m+1} L_{n+1} \quad , \tag{9}$$

which are found in [2; Section 5]. Finally, it is convenient to define $s(n) = n^2 - 3[n^2/3]$, where [ ] denotes the greatest integer function. Since $s(n) = 1$ if $3 \nmid n$ while $s(n) = 0$ if $3 | n$, it follows that

$$(-1)^{s(n)} = (-1)^{F_n} = (-1)^{L_n} .$$

First consider the sequence $Y_n = F_{H_n}$, where H_n obeys $H_{n+2} = H_{n+1} + H_n$. Then using (8), (7), and (5), we find

$$Y_{n+2} = F_{H_{n+2}} = F_{H_{n+1}+H_n} = F_{H_{n+1}-1}F_{H_n} + F_{H_{n+1}}F_{H_n+1}$$

$$= \tfrac{1}{2}F_{H_n}\left(\sqrt{5F^2_{H_{n+1}} + 4(-1)^{H_{n+1}}} - F_{H_{n+1}}\right) + \tfrac{1}{2}F_{H_{n+1}}\left(\sqrt{5F^2_{H_n} + 4(-1)^{H_n}} + F_{H_n}\right)$$

$$= \tfrac{1}{2}\left[Y_n\sqrt{5Y^2_{n+1}\ 4(-1)^{H_{n+1}}} + Y_{n+1}\sqrt{5Y^2_n + 4(-1)^{H_n}}\ \right].$$

If $H_n = F_n$, then $Y_n = U_n$ and we have

$$U_{n+2} = \tfrac{1}{2}\left[U_n\sqrt{5U^2_{n+1}+4(-1)^{s(n+1)}} + U_{n+1}\sqrt{5U^2_n + 4(-1)^{s(n)}}\ \right] \quad (n > 0)$$

while if $H_n = L_n$, then $Y_n = V_n$ and we find

$$V_{n+2} = \tfrac{1}{2}\left[V_n\sqrt{5V^2_{n+1}+4(-1)^{s(n+1)}} + V_{n+1}\sqrt{5V^2_n + 4(-1)^{s(n)}}\ \right] \quad (n > 0)$$

Now consider the sequence $Z_n = L_{H_n}$, where H_n is as before. Using (9), (2), (3), and (6), we see

$$Z_{n+2} = L_{H_{n+2}} = L_{H_{n+1}+H_n} = F_{H_{n+1}-1}L_{H_n} + F_{H_{n+1}}L_{H_n+1}$$

$$= \tfrac{1}{2}L_{H_n}L_{H_{n+1}} - \tfrac{1}{2}L_{H_n}F_{H_{n+1}} + F_{H_{n+1}}L_{H_n+1}$$

$$= \tfrac{1}{2}L_{H_n}L_{H_{n+1}} + \tfrac{1}{2}\sqrt{(L^2_{H_{n+1}} - 4(-1)^{H_{n+1}})/5}\sqrt{5(L^2_{H_n} - 4(-1)^{H_n})}$$

$$= \tfrac{1}{2}\left[Z_{n+1}Z_n + \sqrt{(Z^2_{n+1} - 4(-1)^{H_{n+1}})(Z^2_n - 4(-1)^{H_n})}\ \right]$$

Now if $H_n = F_n$, then $Z_n = X_n$ and we get

$$X_{n+2} = \tfrac{1}{2}\left[X_{n+1}X_n + \sqrt{(X^2_{n+1} - 4(-1)^{s(n+1)})(X^2_n - 4(-1)^{s(n)})}\ \right]$$

and if $H_n = L_n$, we have $Z_n = W_n$ and

$$W_{n+2} = \tfrac{1}{2}\left[W_{n+1}W_n + \sqrt{(W^2_{n+1} - 4(-1)^{s(n+1)})(W^2_n - 4(-1)^{s(n)})}\ \right].$$

See page 86 for References.

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