# AN APPLICATION OF UNIFORM DISTRIBUTIONS TO THE FIBONACCI NUMBERS

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Let $\mu_1 = 1$, $\mu_2 = 2$ and $\mu_n = \mu_{n-1} + \mu_{n-2}$ $(n \geq 3)$ be the Fibonacci numbers and let $p_n$ be the number of digits in $\mu_n$. It is known [1] that the number of divisions required to determine $(\mu_{n+1}, \mu_n)$ by the Euclidean Algorithm is $n$. Also, it is shown in the proof of Lamé's theorem [2] that

$$n < \frac{p_n}{\log \xi} + 1, \quad \text{where} \quad \xi = \frac{1 + \sqrt{5}}{2} \quad .$$

A similar argument [1] shows that

$$n > \frac{p_n - 1}{\log \xi} \quad .$$

Combining these results, we have

$$\left[\frac{p_n - 1}{\log \xi}\right] \leq n - 1 \leq \left[\frac{p_n}{\log \xi}\right] \quad . \tag{1}$$

It has been shown by Brown [3] that the upper bound in (1) is attained for infinitely many $n$. The object of this note is to show that both the upper and lower bounds in (1) are attained for sets of values of $n$ having positive density.

Let $\phi_n$ be the fractional part (mantissa) of $\log \mu_n$. Then, since $p_n = 1 + [\log \mu_n]$, we have $p_n = 1 + \log \mu_n - \phi_n$. Also, since

$$\mu_n \sim \frac{\xi^{n+1}}{\sqrt{5}} \quad ,$$

we have

$$P_n = 1 + (n + 1) \log \xi - \log \sqrt{5} - \phi_n + o(1). \tag{2}$$

Hence

$$n - 1 = \frac{p_n - 1}{\log \xi} - 2 + \frac{\log \sqrt{5}}{\log \xi} + \frac{\phi_n}{\log \xi} + o(1)$$

and it follows that

$$n - 1 < \frac{p_n - 1}{\log \xi} - \frac{1}{4} + 5\phi_n$$

for all sufficiently large n. Thus

$$n - 1 \le \left[\frac{p_n - 1}{\log \xi}\right]$$

and

$$n - 1 = \left[\frac{p_n - 1}{\log \xi}\right]$$

if

$$\phi_n \le \frac{1}{20}$$

and n is sufficiently large.

It also follows from (2) that

$$p_n \le (n + 1) \log \xi - \log \sqrt{5} + \frac{1}{10} + o(1)$$

or

$$n - 1 \ge \frac{p_n}{\log \xi} + \frac{\log \sqrt{5} - \frac{1}{10}}{\log \xi} - 2 + o(1)$$

when

$$\phi_n \geq \frac{9}{10} .$$

Hence

$$n - 1 > \frac{p_n}{\log \xi} - 1$$

and it follows that

$$n - 1 \geq \left[ \frac{p_n}{\log \xi} \right]$$

and

$$n - 1 = \left[ \frac{p_n}{\log \xi} \right]$$

when

$$\phi_n \geq \frac{9}{10}$$

and $n$ is sufficiently large.

The desired result will follow when we show that the sequence $\{\log \mu_n\}$ is uniformly distributed modulo one [4]. By (2) we have

$$\log \mu_n = (n + 1) \log \xi - \log \sqrt{5} + o(1) .$$

Thus, for every positive integer $h$,

$$\begin{aligned} \exp(2\pi ih \log \mu_n) &= \exp(-2\pi ih \log \sqrt{5}) \exp(o(1)) \exp(2\pi ih (n + 1) \log \xi) \\ &= c(1 + o(1)) \exp(2\pi ih (n + 1) \log \xi) , \end{aligned}$$

where c is a constant.

Hence

$$\sum_{n=1}^{m} \exp(2\pi ih \log \mu_n) = c \sum_{n=1}^{m} \exp(2\pi ih(n+1)\log \xi) + o(m) .$$

since $\log \xi$ is irrational, the sequence $\{(n+1)\log \xi\}$ is uniformly distributed modulo one and it follows from Weyl's criterion that the sequence $\{\log \mu_n\}$ is uniformly distributed modulo one.

## REFERENCES


1. R. L. Duncan, "Note on the Euclidean Algorithm," The Fibonacci Quarterly, Vol. 4, No. 4, pp. 367-368.

2. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill, 1939, pp. 43-45.

3. J. L. Brown, Jr., "On Lamé's Theorem," The Fibonacci Quarterly, Vol. 5, No. 2, pp. 153-160

4. Ivan Niven, Irrational Numbers, Carus Monograph No. 11, M. A. A., 1956, Chapter 6.


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