ELEMENTARY PROBLEMS AND SOLUTIONS

Edited By
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

B-112 Proposed by Gerald Edgar, Boulder, Colorado

Let f_n be the generalized Fibonacci sequence (a,b), i.e., $f_1 = a$, $f_2 = b$, and $f_{n+1} = f_n + f_{n-1}$. Let g_n be the associated generalized Lucas sequence defined by $g_n = f_{n-1} + f_{n+1}$. Prove that $f_n g_n = bf_{2n-1} + af_{2n-2}$.

B-113 Proposed by Douglas Lind, Univ. of Virginia, Charlottesville, Va.

Let (x) denote the fractional part of x, so that if $[x]$ is the greatest integer in x, $(x) = x - [x]$. Let $a = (1 + \sqrt{5})/2$ and let A be the set $\{(a), (a^2), (a^3), \ldots\}$. Find all the cluster points of A.

B-114 Proposed by Gloria C. Padilla, Univ. of New Mexico, Albuquerque, New Mexico

Solve the division alphametic

$$\begin{array}{r} \text{PISA} \\ \text{FIB}\overline{\big)\text{ONACCI}} \end{array}$$

where each letter is one of the digits 1, 2, ..., 9 and two letters may represent the same digit. (This is suggested by Maxey Brooke's B-80.)

B-115 Proposed by H. H. Ferns, Victoria, B.C., Canada

From the formulas of B-106:

$$2F_{i+j} = F_iL_j + F_jL_i$$

$$2L_{i+j} = 5F_iF_j + L_iL_j$$

one has

$$F_{2n} = F_nL_n$$

$$F_{3n} = (5F_n^3 + 3F_nL_n^2)/4$$

$$L_{2n} = (5F_n^2 + L_n^2)/2$$

$$L_{3n} = (15F_n^2L_n + L_n^3)/4$$

Find and prove the general formulas of these types.

B-116 Proposed by L. Carlitz, Duke University, Durham, N. Carolina

Find a compact sum for the series

$$\sum_{m,n=0}^{\infty} F_{2m-2n}x^m y^n$$

B-117 Proposed by L. Carlitz, Duke University, Durham, N. Carolina

Find a compact sum for the series

$$\sum_{m,n=0}^{\infty} F_{2m-2n+1}x^m y^n .$$

SOLUTIONS

TERMS OF A DETERMINANT

B-94 Proposed by Clyde A. Bridger, Springfield Jr. College, Springfield, Ill.

Show that the number N_n of non-zero terms in the expansion of

$$K_n = \begin{vmatrix} a_1 & b_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & a_2 & b_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & a_3 & b_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots \\ \cdots \\ \cdots \\ 0 & 0 & 0 & \cdots & 0 & -1 & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 & a_n \end{vmatrix}$$

is obtained by replacing each a_i and each b_i by 1 and evaluating K_n. Show further that $N_n = F_{n+1}$, the $(n + 1)$st Fibonacci number.

Solution by F. D. Parker, St. Lawrence University, Canton, N.Y.

Expanding by the last column, we have $K_n = a_n K_{n-1} + b_{n-1} K_{n-2}$. Hence, if N_n is the number of non-zero terms in the expansion, we have $N_n = N_{n-1} + N_{n-2}$. But $N_1 = 1$, $N_2 = 2$, so that $N_n = F_{n+1}$.

Also solved by M.N.S. Swamy and the proposer.

A FIBONACCI FACTORIAL

B-95 Proposed by Brother U. Alfred, St. Mary's College, California.

What is the highest power of 2 that exactly divides

$$F_1F_2F_3 \ldots F_{100} \quad ?$$

Solution by Charles W. Trigg, San Diego, California.

For $n \geq 3$, F_k is divisible by 2^n if k is of the form $2^{n-2} \cdot 3(1+2m)$, F_k is divisible by 2 but by no higher power of 2. Hence, the highest power of 2 that exactly divides $F_1F_2F_3 \ldots F_{100}$ is

$$[(100 - 3(/6 + 1] + 3[(100 + 6(/12] + 4[112/24] + 5[124/48]$$
$$+ 6[148/96] + 7[196/192] \text{ or } 80.$$

As usual, $[x]$ indicates the largest integer in x.

Also solved by Sidney Kravitz, Dewey C. Duncan, and the proposer.

Editorial note: The results in the above solution indicate that the answer may also be expressed as

$$[100/3] + 2[100/6] + [100/12] + [100/24] + [100/48]$$
$$+ [100/96] = 33 + 32 + 8 + 4 + 2 + 1 = 80.$$

LIMITED PARTITIONS

B-96 Proposed by Phil Mana, Univ. of New Mexico, Albuquerque, New Mex.

Let G_n be the number of ways of expressing the positive integer n as an ordered sum $a_1 + a_2 + \ldots + a_s$ with each a_i in the set 1, 2, 3. (For example, $G_3 = 4$ since 3 has just the expressions $3, 2 + 1, 1 + 2, 1 + 1 + 1$.) Find and prove the lowest order linear homogeneous recursion relation satisfied by the G_n.

Solution by the proposer.

Removing the forst term a_1 (which is 1, 2, or 3) from all allowable sums for an $n > 3$ gives all allowable sums for n-1, n-2, and n-3 in unique fashion. Hence $G_n = G_{n-1} + G_{n-2} + G_{n-3}$ for $n > 3$. There is no lower order linear homogeneous recursion relation for the G_n since

$$\begin{vmatrix} G_1 & G_2 & G_3 \\ G_2 & G_3 & G_4 \\ G_3 & G_4 & G_5 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 \\ 2 & 4 & 7 \\ 4 & 7 & 13 \end{vmatrix} \neq 0.$$

DENSITY OF THE FIBONACCI NUMBERS

B-97 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let $A = \{a_k\}_{k=1}^{\infty}$ be an increasing sequence of numbers and let A(n) denote the number of terms of A not greater than n. The Schnirelmann density of A is defined as the greatest lower bound of the ratios A(n)/n for n=1, 2, ... Show that the Fibonacci sequence has density zero.

Solution by the proposer.

Let $a = (1 + \sqrt{5})/2$, and $F = \{F_n\}_{n=2}^{\infty}$ be the Fibonacci sequence. It is easy to show by indiction that $a^{n-2} < F_n$ for $n > 0$, so that $F(n) < \log_a(n+2)$. Then since $0 \le F(n)/n$,

$$0 \le \lim_{n\to\infty} \frac{F(n)}{n} \le \lim_{n\to\infty} \frac{\log_a(n+2)}{n} = 0,$$

so that the density of F is 0.

Also solved by C.B.A. Peck.

A COMPACT FINITE GENERATING FUNCTION

B-98 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let F_n be the n^{th} Fibonacci number and find a compact expression for the sum

$$S_n(x) = F_1x^2 + F_3x^3 + \ldots + F_nx^n.$$

Solution by Gloria C. Padilla, University of New Mexico, Albuquerque, N.M.

One easily sees that

$$(x^2 + x - 1)\, S_n(x) = -x + (F_{n-1} + F_n)\, x^{n+1} + F_nx^{n+2}.$$

Hence

$$S_n(x) = (-x + F_{n+1}x^{n+1} + F_nx^{n+2})/(x^2 + x - 1).$$

Also solved by L. Carlitz, Dewey C. Duncan, F. D. Parker, M. N. S. Swamy, Howard L. Walton, David Zeitlin (who pointed out that the result is a special case of formula (5) of his paper "On summation formula for Fibonacci and Lucas numbers" this Quarterly, Vol. 2, No. 2, 1964, p. 105), and the proposer.

COMPACT INFINITE SUM

B-99 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

$$T(x) = S_1(x) + \frac{S_2(x)}{2\,!} + \frac{S_2(x)}{3\,!} + \ldots ,$$

where $S_n(x)$ is as defined in B-98.

Solution by David Zeitlin, Minneapolis, Minnesota.

From B-98, we obtain

$$(1 - x - x^2)\, T(x) = -x^2 \sum_{n=0}^{\infty} \frac{F_n x^n}{n!}$$

$$- \sum_{n=0}^{\infty} \frac{(n+1)\, F_{n+1} x^{n+1}}{(n+1)!} + ex.$$

Let a and b be the roots of

$$x^2 - x - 1 = 0.$$

Then

$$\frac{e^{ax} - e^{bx}}{a-b} = \sum_{n=0}^{\infty} \frac{F_n x^n}{n!},$$

and

$$\frac{x(ae^{ax} - be^{bx})}{a-b} = \sum_{n=0}^{\infty} \frac{(n+1)\, F_{n+1} x^{n+1}}{(n+1)!},$$

Thus,

$$(1 - x - x^2)\, T(x) = -x^2\left(\frac{e^{ax} - e^{bx}}{a-b}\right) - x\left(\frac{ae^{ax} - be^{bx}}{a-b}\right) + ex.$$

Also solved by L. Carlitz, Dewey C. Duncan, M.N.S. Swamy, and the proposer.

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