RECURRENCE RELATIONS FOR SEQUENCES LIKE $\{F_{F_n}\}$

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In [1] the problem of finding recurrence relations for the sequences $\{F_{F_n}\}$, $\{F_{L_n}\}$, $\{L_{F_n}\}$, $\{L_{L_n}\}$ — where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively — is proposed. What follows is an investigation of this problem and some of its generalizations.

Let r and s be any two nonzero elements of a field $F^* = (F, +, \cdot)$ in which r^n is defined in the usual way with the field operations, $+, \cdot$. Define $\{U_n\}$ and $\{V_n\}$ by $U_n = (r^n - s^n)/(r - s)$ and $V_n = r^n + s^n$ for all integers n. Furthermore, let $\{H_n\}$ be any generalized Fibonacci sequence consisting of integers — that is H_0 and H_1 are integers and $H_{n+2} = H_{n+1} + H_n$ for all integers n. Some recurrence relations for sequences such as $\{U_{H_n}\}$ and $\{V_{H_n}\}$ will be derived here.

Let $\{g_n\}$ be any sequence in n obeying the recurrence relation $g_{n+2} = (r + s)g_{n+1} - rsg_n$ for all integers n. Then there are constants C_1 and C_2 in F^* such that $g_n = C_1r^n + C_2s^n$ for all integers n. Define $\{X_n\}$, $\{Y_n\}$ and $\{G_n\}$ by $X_n = U_{H_n}$, $Y_n = V_{H_n}$ and $G_n = g_{H_n}$ for all integers n. From here on, when n is written, understand that n can take on all integer values unless otherwise indicated. For convenience write $R_n = r^{H_n}$ and $S_n = s^{H_n}$.

Consider the product $G_{n+2}Y_{n+1}$.

$$\begin{aligned} G_{n+2}Y_{n+1} &= (C_1R_{n+2} + C_2S_{n+2})(R_{n+1} + S_{n+1}) \\ &= C_1R_{n+2}R_{n+1} + C_2S_{n+2}S_{n+1} + C_1R_{n+2}S_{n+1} + C_2R_{n+1}S_{n+2} \\ &= C_1R_{n+3} + C_2S_{n+3} + R_{n+1}S_{n+1}(C_1R_n + C_2S_n) \\ &= G_{n+3} + (rs)^{H_{n+1}} \cdot G_n \end{aligned}$$

Thus,

$$(1) \qquad G_{n+3} = G_{n+2}Y_{n+1} - (rs)^{H_{n+1}}G_n \quad .$$

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A corollary to (1) is the relatively simple recurrence relation for $\{Y_n\}$.

(2) $$Y_{n+3} = Y_{n+2}Y_{n+1} - (rs)^{H_{n+1}} \cdot Y_n \quad .$$

When $rs = \pm 1$, (2) is especially simple;

$$r = \frac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad s = \frac{1}{2}(1 - \sqrt{5})$$

gives

(3) $$L_{H_{n+3}} = L_{H_{n+2}}L_{H_{n+1}} - (-1)^{H_{n+1}}L_{H_n} \quad ,$$

where L_n is the n^{th} Lucas number.

Consider the product $Y_{n+2}G_{n+1}$.

$$\begin{aligned} Y_{n+2}G_{n+1} &= C_1R_{n+2}R_{n+1} + C_2S_{n+2}S_{n+1} + C_1R_{n+1}S_{n+2} + C_2R_{n+2}S_{n+1} \\ &= G_{n+3} + R_{n+1}S_{n+1}(C_1S_n + C_2R_n) \quad . \end{aligned}$$

But

$$C_1s^n + C_2r^n = (C_1 + C_2)V_n - (C_1r^n + C_2s^n) = g_0V_n - g_n \quad .$$

Thus

$$C_1S_n + C_2R_n = g_0Y_n - G_n \quad ,$$

and

$$Y_{n+2}\,G_{n+1} = G_{n+3} + (rs)^{H_{n+1}}(g_0\,Y_n - G_n) \quad .$$

That is,

(4) $$G_{n+3} = Y_{n+2}G_{n+1} + (rs)^{H_{n+1}}(G_n - g_0Y_n) \quad .$$

Add (1) and (4) to get

$$(5) \qquad 2G_{n+3} = G_{n+1}Y_{n+2} + G_{n+2}Y_{n+1} - (rs)^{H_{n+1}}g_0Y_n$$

Now consider the product $(r - s)^2X_{n+2}Y_{n+1}$.

$$\begin{aligned}(r - s)^2X_{n+1}X_{n+2} &= (R_{n+2} - S_{n+2})(R_{n+1} - S_{n+1}) \\ &= R_{n+3} + S_{n+3} - R_{n+1}S_{n+1}(R_n + S_n) \\ &= Y_{n+3} - (rs)^{H_{n+1}}Y_n .\end{aligned}$$

Thus,

$$(6) \qquad Y_{n+3} = (r - s)^2X_{n+2}X_{n+1} + (rs)^{H_{n+1}}Y_n .$$

Some second-order recurrence relations can be obtained by using the following simple and easily verified identities — which hold for all integers a and b — by putting $a = H_n$ and $b = H_{n+1}$ or $a = H_{n+1}$ and $b = H_n$.

$$U_{a+b} = r^aU_b + s^bU_a$$

$$V_{a+b} = r^aV_b - (r - s)s^bU_a = s^aV_b + (r - s)r^bU_a$$

$$(r - s)U_{a+b} = r^aV_b - s^bV_a$$

Some of the recurrence relations are

$$(7) \qquad \begin{aligned} X_{n+2} &= R_nX_{n+1} + S_{n+1}X_n = S_nX_{n+1} + R_{n+1}X_n \\ Y_{n+2} &= R_nY_{n+1} - (r - s)S_{n+1}X_n \\ &= S_nY_{n+1} + (r - s)R_{n+1}X_n \\ &= (r - s)R_nX_{n+1} + S_{n+1}Y_n \\ &= -(r - s)S_nX_{n+1} + R_{n+1}Y_n \end{aligned}$$

From (7) it immediately follows that

(7')
$$2X_{n+2} = X_{n+1}Y_n - X_nY_{n+1}$$
$$2Y_{n+2} = (r - s)^2X_{n+1}X_n - Y_{n+1}Y_n$$

For a fixed integer j define $\{Z_n\}$ and $\{W_n\}$ by $Z_n = U_{H_n+j}$ and $W_n = V_{H_n+j}$. Thus,

$$(r - s)Z_n = r^jR_n - s^jS_n \quad \text{and} \quad W_n = r^jR_n + s^jS_n \ .$$

Now,

$$\begin{aligned}(r - s)Z_{n+2} &= r^jR_{n+1}R_n - s^jS_{n+1}S_n \\ &= R_n(r^jR_{n+1} - s^jS_{n+1}) + S_{n+1}(r^jR_n - s^jS_n) \\ &\qquad - R_nS_{n+1}(r^j - s^j)\end{aligned}$$

so that

(8)
$$Z_{n+2} = R_nZ_{n+1} + S_{n+1}Z_n - (rs)^{H_n}S_{n-1}U_j$$

Similarly,

(9)
$$Z_{n+2} = S_nZ_{n+1} + R_{n+1}Z_n - (rs)^{H_n}R_{n-1}U_j$$

Add (8) and (9) to get

(10)
$$2Z_{n+2} = Y_nZ_{n+1} + Y_{n+1}Z_n - (rs)^{H_n}Y_{n-1}U_j$$

Also,

$$\begin{aligned}W_{n+2} &= r^jR_{n+1}R_n + s^jS_{n+1}S_n \\ &= R_n(r^jR_{n+1} - s^jS_{n+1}) + S_{n+1}(r^jR_n + s^jS_n) \\ &\qquad - R_nS_{n+1}(r^j - s^j)\end{aligned}$$

and

$$(11) \qquad W_{n+2} = (r - s)R_n Z_{n+1} + S_{n+1} W_n - (r - s)(rs)^{H_n} S_{n-1} U_j \;;$$

Similarly,

$$(12) \qquad W_{n+2} = (s - r)S_n Z_{n+1} + R_{n+1} W_n - (s - r)(rs)^{H_n} R_{n-1} U_j$$

Add (11) and (12) to get

$$(13) \qquad 2W_{n+2} = (r - s)^2 X_n Z_{n+1} + Y_{n+1} W_n + (r - s)^2 (rs)^{H_n} X_{n-1} U_j \;.$$

When $r = (1 + \sqrt{5})/2$ and $s = (1 - \sqrt{5})/2$, (10) and (13) become

$$(14) \qquad \begin{aligned} 2F_{H_{n+2}+j} &= L_{H_n} F_{H_{n+1}+j} + L_{H_{n+1}} F_{H_n+j} - (-1)^{H_n} L_{H_{n-1}} F_j \\ 2L_{H_{n+2}+j} &= 5F_{H_n} F_{H_{n+1}+j} + L_{H_{n+1}} L_{H_n+j} + 5(-1)^{H_n} F_{H_{n-1}} F_j \end{aligned}$$

where F_n is the n^{th} Fibonacci number and L_n is the n^{th} Lucas number.

The techniques used above in deriving recurrence relations are not entirely inhibited when sequences of the type $\{U_{K_n}\}$ and $\{V_{K_n}\}$, where $\{K_n\}$ is a sequence of integers obeying a linear, homogeneous recurrence relation with constant coefficients, are considered. Let $\{K_n\}$ obey the recurrence relation

$$K_{n+m} = \sum_{j=0}^{m} p_j K_{n+m-j} \quad ,$$

where m is a fixed integer, and with p_j, K_n being integers when n is nonnegative. Then $\{V_{K_n}\}$ and $\{U_{K_n}\}$ are defined for n nonnegative; if $p_m = \pm 1$, then the definition applies for all n. Repeated application of the identity

$U_{a+b} = r^a U_b + s^b U_a$ gives $U_{a_1+a_2+\cdots a_m}$ as a linear combination of U_{aj}, $j = 1, 2, \cdots, m$, with the coefficients being products of powers of r and s. By putting $a_j = p_j K_{n+m-j}$, when $n + m - j$ is nonnegative, and by utilizing repeatedly the identities

$$U_{-n} = -(rs)^{-n} U_n$$

$$U_{2n} = U_n V_n$$

$$U_{(2k+1)n} = U_n\left[(rs)^{kn} + \sum_{j=0}^{k-1} (rs)^{jn} V_{2(k-j)n}\right], \quad k \geq 1,$$

m^{th} order recurrence relations are easily produced for $\{U_{K_n}\}$. $\{V_{K_n}\}$ may be treated similarly by repeated application of the identity $V_{a+b} = r^a V_b - (r - s)s^b U_a$ and by utilization of the identities

$$V_{-n} = (rs)^{-n} V_n$$

$$V_{2kn} = V_{kn}^2 - 2(rs)^{kn}$$

$$V_{(2k+1)n} = V_n(-r^n s^n)^k + \sum_{j=0}^{k-1} (-r^n s^n)^j V_{2(k-j)n}, \quad k \geq 1$$

$$(r - s)U_{a+b} = r^a V_b - s^b V_a .$$

A special case of interest occurs when $m = 3$ and $p_j = 1$, $j = 1, 2, 3$. Letting $A_n = r^{K_n}$ and $B_n = s^{K_n}$, $D_n = U_{K_n}$ and $E_n = V_{K_n}$, then $U_{a+b} = r^a U_b - s^b U_a$ gives

$$D_{n+3} = A_n A_{n+1} D_{n+2} + A_n B_{n+2} D_{n+1} + B_{n+1} B_{n+2} D_n \tag{15}$$
$$= A_n A_{n+1} D_{n+2} + B_n B_{n+2} D_{n+1} + A_{n+1} B_{n+2} D_n$$

and

$$2D_{n+3} = 2A_n A_{n+1} D_{n+2} + B_{n+2} E_n D_{n+1} + B_{n+2} E_{n+1} D_n .$$

Similarly,

$$2D_{n+3} = 2B_nB_{n+1}D_{n+2} + A_{n+2}E_nD_{n+1} + A_{n+2}E_{n+1}D_n .$$

Thus,

$$4D_{n+3} = 2(A_nA_{n+1} + B_nB_{n+1})D_{n+2} + E_nE_{n+2}D_{n+1} + E_{n+1}E_{n+2}D_n .$$

But

$$\begin{aligned} A_nA_{n+1} + B_nB_{n+1} &= A_n(A_{n+1} + B_{n+1}) - B_{n+1}(A_n - B_n) \\ &= A_nE_{n+1} - B_{n+1}(r - s)D_n \\ &= B_nE_{n+1} + A_{n+1}(r - s)D_n , \end{aligned}$$

so that

$$2(A_nA_{n+1} + B_nB_{n+1}) = E_nE_{n+1} + (r - s)^2D_nD_{n+1} ,$$

and

$$(16) \quad 4D_{n+3} = (E_nE_{n+1} + (r - s)^2D_nD_{n+1})D_{n+2} + E_nE_{n+2}D_{n+1} + E_{n+1}E_{n+2}D_n .$$

Also, $V_{a+b} = r^aV_b - (r - s)s^bU_a$ and $(r - s)U_{a+b} = r^aV_b - s^bV_a$ give

$$(17) \quad \begin{aligned} E_{n+3} &= A_nA_{n+1}E_{n+2} - A_nB_{n+2}E_{n+1} + B_{n+1}B_{n+2}E_n \\ &= A_nA_{n+1}E_{n+2} + B_nB_{n+2}E_{n+1} - A_{n+1}B_{n+2}E_n \end{aligned}$$

and

$$2E_{n+3} = 2A_nA_{n+1}E_{n+2} - (r - s)B_{n+2}D_nE_{n+1} - (r - s)B_{n+2}D_{n+1}E_n .$$

Similarly,

$$2E_{n+3} = 2B_nB_{n+1}E_{n+2} + (r - s)A_{n+2}D_nE_{n+1} + (r - s)A_{n+2}D_{n+1}E_n .$$

Thus,

4E $$4E_{n+3} = 2(A_nA_{n+1} + B_nB_{n+1})E_{n+2} + (r - s)^2D_nD_{n+2}E_{n+1} + (r - s)^2D_{n+1}D_{n+2}E_n$$

and

$$4E_{n+3} = (E_nE_{n+1} + (r - s)^2D_nD_{n+1})E_{n+2} + (r - s)^2D_nD_{n+2}E_{n+1} + (r - s)^2D_{n+1}D_{n+2}E_n \quad (18)$$

Given D_0, D_1, D_2, E_0, E_1 and E_2, (16) and (18) completely determine $\{D_n\}$ and $\{E_n\}$, for $n \geq 0$.

REFERENCES

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2. Whitney, R., "Composition of Recursive Formulae," Fibonacci Quarterly, Vol. 4, No. 4, pp. 363-366.

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