# THE HEIGHTS OF FIBONACCI POLYNOMIALS AND AN ASSOCIATED FUNCTION

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Define the sequence of Fibonacci polynomials $\{f_n(x)\}$ by

$$f_1(x) = 1, \quad f_2(x) = x; \quad f_n(x) = xf_{n-1}(x) + f_{n-2}(x) \quad (n \geq 3) .$$

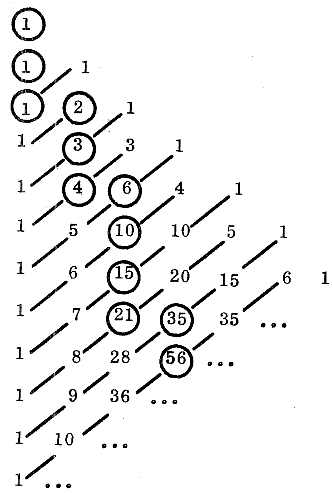
Then it has been shown [3] that

$$(1) \qquad f_n(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-j-1}{j} x^{n-2j-1} ,$$

where $[x]$ represents the greatest integer contained in x. Since $f_{n+1}(x) = i^{-n}U_n(ix/2)$, where the $U_n(x)$ are the Chebyshev polynomials of the second kind, we note that the Fibonacci polynomials are essentially the Chebyshev polynomials. Define the Fibonacci sequence $\{F_n\}$ by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ $(n \geq 3)$, and the Lucas sequence $\{L_n\}$ by $L_1 = 1$, $L_2 = 3$, $L_n = L_{n-1} + L_{n-2}$ $(n \geq 3)$. It then follows from these recurrence relations that $f_n(1) = F_n$. By the height of $f_n(x)$, denoted by $m(n)$, we mean the greatest coefficient of $f_n(x)$, that is

$$m(n) = \max \left\{ \binom{n-j-1}{j} \right\} \qquad (j = 0, 1, \cdots, [(n-1)/2]) .$$

Since the coefficients in (1) are diagonals of Pascal's Triangle, the $m(n)$ are the maximum entries along these diagonals, and they form the pattern exhibited below. Interest was first aroused in these numbers when it was observed that if the heights $m(n)$ and $m(n+1)$ were in adjacent columns, then they were in the ratio of consecutive Fibonacci numbers (e.g., 1:2, 4:6 = 2:3, 21:35 = 3:5). Although this is not true in general, some interesting properties derived from these ratios were found.




In order to define these cross-over ratios, we must first verify that this initial pattern continues, so that the only changes in the column pattern are lateral jumps of one column. We do this in the following:

Theorem 1. Denote logical implication by "$\Rightarrow$". Then

(i) $$\binom{n}{k} \ge \binom{n-1}{k+1} \Rightarrow \binom{n-1}{k+1} \ge \binom{n-2}{k+2},$$

(ii) $$\binom{n-1}{k+1} > \binom{n}{k} \Rightarrow \binom{n}{k+1} > \binom{n+1}{k},$$

(iii) $$\binom{n}{k} > \binom{n+1}{k-1} \Rightarrow \binom{n+1}{k=1} > \binom{n+2}{k-1}$$

(iv) $$\binom{n}{k} > \binom{n-1}{k+1} \text{ and } \binom{n+1}{k} \le \binom{n}{k+1} \Longrightarrow \binom{n}{k+1} > \binom{n-1}{k+2}$$

Proof. We prove (i), the other parts using similar techniques. (i) is trivial for $n \le k+1$. Assume $n - k \ge 2$, so that denoting logical equivalence by "$\Leftrightarrow$" we have

$$\binom{n}{k} \ge \binom{n-1}{k+1} \Leftrightarrow n(k+1) \ge (n-k)(n-k-1) \Rightarrow n(k+1) + n - k - 2 = (n-1)(k+2)$$

$$\ge (n-k)^2 - 2 + 8 - 5(n-k) = (n-k-2)(n-k-3) \Longleftrightarrow \binom{n-1}{k+1} \ge \binom{n-2}{k+2}.$$

A little reflection will show that these results imply that if

$$m(n) = \binom{u}{v}, \quad \text{then } m(n+1) = \binom{u+1}{v} \text{ or } \binom{u}{v+1} .$$

Call the column of ones the $0^{th}$ column, and label the other columns of Pascal's Triangle consecutively. Choose $n$ such that $m(n)$ appears in the $(k-1)^{th}$ column and $m(n+1)$ appears in the $k^{th}$ column. Then $r_k = m(n)/m(n+1)$ is called the $k^{th}$ cross-over ratio. By Theorem 1, $r_k$ is well-defined and unique.

Theorem 2. For $r_k$ as defined above we have

$$r_k = k/[\tfrac{1}{2}(k + 1 + \sqrt{5k^2 - 2k + 1})] .$$

where $[x]$ denotes the greatest integer contained in $x$.

Proof. If $n$ is the greatest integer for which

$$\binom{n}{k-1} > \binom{n-1}{k} ,$$

then clearly

$$\binom{n}{k-1}$$

is the greatest height in the $(k-1)^{th}$ column. This criterion is equivalent to

$$nk > (n-k-1)(n-k) \Longleftrightarrow n^2 - (3k-1)n + k^2 - k < 0 .$$

The greatest $n$ for which this holds is the greatest integer contained in the largest root of

$$n^2 - (3k-1)n + k^2 - k = 0 .$$

so that

$$n = [\tfrac{1}{2}(3k - 1 + \sqrt{5k^2 - 2k + 1})] .$$

Thus

$$r_k = \binom{n}{k-1} \Big/ \binom{n}{k} = \frac{k}{n - k + 1} = k/[\tfrac{1}{2}(k + 1 + \sqrt{5k^2 - 2k + 1})].$$

This result makes computation of the cross-over ratio for a given column simple. A limited evaluation of the expression in the denominator is given later in the paper. From Theorem 2 we may conclude

Theorem 3. For $\alpha = (1 + \sqrt{5})/2$, we have

$$\lim_{n \to \infty} r_n = 1/\alpha .$$

Proof. Since

$$n/\{\tfrac{1}{2}(n + 1 + \sqrt{5n^2 - 2n + 1})\} \le r_n < n/\{\tfrac{1}{2}(n - 1 + \sqrt{5n^2 - 2n + 1})\} ,$$

the result follows from

$$\lim_{n \to \infty} n/\{\tfrac{1}{2}(n + 1 + \sqrt{5n^2 - 2n + 1})\} = \lim_{n \to \infty} n/\{\tfrac{1}{2}(n - 1 + \sqrt{5n^2 - 2n + 1})\} = 1/\alpha$$

It has been shown [4] that

$$\lim_{n \to \infty} F_n/F_{n+1} = \lim_{n \to \infty} L_n/L_{n+1} = 1/\alpha ,$$

so it is not surprising to observe that $F_n/F_{n+1}$ is a cross-over ratio for $n \ge 2$, and $L_n/L_{n+1}$ is one for $n \ge 4$. It is our aim to prove this holds in general.

Theorem 4. Let $h(k) = [\tfrac{1}{2}(k + 1 + \sqrt{5k^2 - 2k + 1})]$. Then $h(F_n) = F_{n+1}$ for $n \ge 2$, and $h(L_n) = L_{n+1}$ for $n \ge 4$.

Proof. We will prove $h(F_n) = F_{n+1}$, the proof for Lucas numbers involving no new ideas. Since $x - 1 < [x] \le x$, the assertion is equivalent to

$$\tfrac{1}{2}(F_n - 1 + \sqrt{5F_n^2 - 2F_n + 1}) < F_{n+1} \le \tfrac{1}{2}(F_n + 1 + \sqrt{5F_n^2 - 2F_n + 1}) .$$

The right side is equivalent to

$$(2F_{n+1} - F_n - 1)^2 = (L_n - 1)^2 \le 5F_n^2 - 2F_n + 1$$

$$\Longleftrightarrow L_n^2 - 5F_n^2 \le 2(L_n - F_n) = 4F_{n-1} .$$

Now it is known [2; Identity XI] that $L_n^2 - 5F_n^2 = 4(-1)^n$, so the last inequality is valid for $n > 2$. The case $n = 2$ is verified directly. The left side is equivalent to

$$(2F_{n+1} - F_n + 1)^2 = (L_n + 1)^2 > 5F_n^2 - 2F_n + 1$$

$$\Longleftrightarrow L_n^2 - 5F_n^2 > -2(L_n + F_n)$$

$$\Longleftrightarrow 4(-1)^n > -4F_{n+1}$$

which is valid for $n > 1$, completing the proof.

<u>Theorem 5.</u> $F_n/F_{n+1}$ is a cross-over ratio for $n \ge 2$, and $L_n/L_{n+1}$ is a cross-over ratio for $n \ge 4$.

<u>Proof.</u> By Theorems 2 and 4 we have $r_{F_n} = F_n/F_{n+1}$ for $n \ge 2$, and $r_{L_n} = L_n/L_{n+1}$ for $n \ge 4$.

We mention in passing that the results of Theorem 4,

$$F_{n+1} = [\tfrac{1}{2}(F_n + 1 + \sqrt{5F_n^2 - 2F_n + 1})] \qquad (n \ge 2)$$

$$L_{n+1} = [\tfrac{1}{2}(L_n + 1 + \sqrt{5L_n^2 - 2L_n + 1})] \qquad (n \ge 4) ,$$

form an essentially different solution to Problem B-42 [1], perhaps an improvement over the published solution since the value of n is not required.

We shall apply these results in a test to determine whether a given integer is a Fibonacci number, but we need first to establish a certain property of Fibonacci-type sequences.

Theorem 6. Define a Fibonacci sequence $\{f_n\}$ by specifying two integers $f_0$ and $f_1$, along with the recurrence relation $f_n = f_{n-1} + f_{n-2}$. Then

$$\left|f_{n+1}^2 - f_{n+1}f_n - f_n^2\right| = \left|f_1^2 - f_1f_0 - f_0^2\right| = D$$

for all $n \geq 0$.

Proof. We proceed by induction. The statement is true for $n = 0$. Assume it is true for $n = k \geq 0$. Then

$$D = \left|f_{k+1}^2 - f_{k+1}f_k - f_k^2\right| = \left|-f_{k+1}^2 + f_{k+1}f_k + f_k^2 + f_{k+1}^2 + f_{k+1}f_k - f_{k+1}^2 - f_{k+1}f_k\right|$$
$$= \left|(f_{k+1} + f_k)^2 - f_{k+1}(f_{k+1} + f_k) - f_{k+1}^2\right| = \left|f_{k+2}^2 - f_{k+2}f_{k+1} - f_{k+1}^2\right| ,$$

so the assertion is true for $n = k + 1$, completing the induction step and the proof.

Now for the Fibonacci sequence $D = |F_1^2 - F_1F_0 - F_0^2| = 1$. Since $h(F_n) = F_{n+1}$, all Fibonacci numbers $F_n$ satisfy

$$\left|h^2(F_n) - F_nh(F_n) - F_n^2\right| = 1 .$$

We shall show that only Fibonacci numbers satisfy this equation, thus providing a necessary and sufficient condition for an integer to be a Fibonacci number.

Theorem 7. Let $m$ be a positive integer, and $g(m) = |h^2(m) - mh(m) - m^2|$. Then $m$ is a Fibonacci number if and only if $g(m) = 1$. Also, $m \geq 7$ is a Lucas number if and only if $g(m) = 5$.

Proof. We have shown above that if $m$ is a Fibonacci number then $g(m) = 1$. Now assume $g(m) = 1$, and we wish to show $m$ is a Fibonacci number. Since $h(m) > m$, we may form a decreasing Fibonacci sequence

(2) $$h(m),\ m,\ h(m) - m,\ 2m - h(m), \cdots,\ f_1,\ f_0 ,$$

where $f_0$ is the least nonnegative term of this sequence. Then $f_0 < \frac{1}{2}f_1$, for if $f_1 > f_0 > \frac{1}{2}f_1$, then there is another term of the sequence $f_{-1}$ such that

$0 \le f_{-1} = f_1 - f_0 < f_0$ contradicting the definition of $f_0$, while if $f_0 = \frac{1}{2}f_1, f_0 - (f_1 - f_0) = 0$ is another term of the sequence $<f_0$, again contradicting the definition of $f_0$. Thus $f_1 = 2f_0 + a$ where $a > 0$. But by Theorem 6, $1 = g(m) = \left|(2f_0 + a)^2 - (2f_0 + a)f_0 - f_0^2\right| = \left|f_0^2 + 3af_0 + a^2\right|$, and since $f_0$ and $a$ are nonnegative integers, we must have $f_0 = 0$, $a = 1$, so that $f_1 = 1$. Hence $m$ is a member of a Fibonacci sequence which begins with $f_0 = 0$ and $f_1 = 1$; that is, $m$ is a Fibonacci number.

We now prove the latter half of the theorem. Suppose $m \ge 7$ is a Lucas number $L_n$. For the Lucas sequence $D = 5$, and so by Theorems 4 and 5 we have $g(m) = 5$. Now assume $g(m) = 5$ where $m \ge 7$, and as above let $f_0$ be the least nonnegative term in a decreasing Fibonacci sequence defined in (2). Clearly $f_0 > 0$, for $f_0 = 0$ implies $5 = g(m) = \left|f_1^2 - f_1f_0 - f_0^2\right| = \left|f_1^2\right.$. Also, as in the first section, $f_0 < \frac{1}{2}f_1$, so $f_1 = 2f_0 + a$ where $a > 0$. Then $5 = g(m) = \left|(2f_0 + a)^2 - (2f_0 + a)f_0 - f_0^2\right| = \left|f_0^2 + 3af_0 + a^2\right|$, and since $f_0$ and $a$ are positive integers, we must have $f_0 = a = 1$, so that $f_1 = 3$. Thus $m$ belongs to a Fibonacci sequence with $f_0 = 1$, $f_1 = 3$; that is, $m$ is a Lucas number.

We note that Theorem 6 is also implied by the result of Long and Jordan [5] that the only solutions of the diophantine equation $\left|x^2 - 5y^2\right| = 4$ are $x = L_n$, $y = F_n$.

Define

$$h_n(k) \quad \text{for} \quad n \ge 0$$

by

$$h_0(k) = k \qquad \text{and} \qquad h_n(k) = h\{h_{n-1}(k)\}$$

for $n \ge 0$. Then

$$\{h_n(k)\}_{n=0}^{\infty}$$

is a sequence of integers for each choice of $k$. Values of $h_n(k)$ for $1 \le k \le 10$ and $0 \le n \le 9$, which were computed by Terry Brennan, are given in Table 1. It appears from the Table that

$$\{h_n(k)\}$$

obeys either a homogeneous or nonhomogeneous Fibonacci recurrence relation. We prove this holds in general.

Table 1

Values of $h_n(k)$

| k \ n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |
| 2 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 |
| 3 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 |
| 4 | 5 | 6 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 |
| 5 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 | 377 |
| 6 | 6 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 | 466 |
| 7 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | 521 |
| 8 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 | 377 | 610 |
| 9 | 9 | 14 | 22 | 35 | 56 | 90 | 145 | 234 | 378 | 611 |
| 10 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 | 466 | 754 |

Theorem 8. For each choice of $k$ the sequence $\{h_n(k)\}$ obeys one of the following recurrence relations:

$$h_{n+2}(k) = h_{n+1}(k) + h_n(k), \quad n = 0, 1, \cdots \quad \text{(Fibonacci homogeneous)}$$

$$h_{n+2}(k) = h_{n+1}(k) + h_n(k) - 1, \quad n = 0, 1, \cdots \quad \text{(Fibonacci nonhomogeneous)}$$

Proof. The assertion is true for $k = 1$ since $h_n(1) = F_{n+2}$ obeys the first relation. We thus consider $k \geq 1$. We shall use the property that $x - 1 \leq [x] \leq x$ to show that $h_2(k) = h_0(k) + h_1(k)$ or $h_0(k) + h_1(k) - 1$. We shall then use induction to prove this initial recurrence continues to hold throughout the sequence. For sake of brevity we let throughout the rest of the paper

$$h_n(k) \equiv h_n, \quad h_0(k) \equiv k, \quad h_1(k) \equiv h(k) \equiv h, \text{ and } "\Leftarrow"$$

mean "if". From the definition we have

$$\tfrac{1}{2}(h_n - 1 + \sqrt{5h_n^2 - 2h_n + 1}) \leq h_{n+1} \leq \tfrac{1}{2}(h_n + 1 + \sqrt{5h_n^2 - 2h_n + 1}) .$$

Then

$$
\begin{aligned}
h_2(k) \geq k + h(k) - 1 &\Leftarrow \tfrac{1}{2}(h - 1 + \sqrt{5h^2 - 2h + 1}) \geq k + h - 2 \\
&\Longleftrightarrow 5h^2 - 2h + 1 \geq (2k + h - 3)^2 \\
&\Longleftrightarrow h^2 \geq k^2 + kh - h - 3k + 2 \\
&\Leftarrow k^2 + \tfrac{1}{2}(k^2 + k + k\sqrt{5k^2 - 2k + 1}) - \tfrac{1}{2}(k - 1 + \\
&\qquad \sqrt{5k^2 - 2k + 1}) - 2k + 2 \leq \tfrac{1}{4}(k - 1 + \\
&\qquad \sqrt{5k^2 - 2k + 1})^2 \\
&\Longleftrightarrow 8 \leq 8k
\end{aligned}
$$

which is valid. Also

$$
\begin{aligned}
h_2(k) \leq k + h(k) &\Longleftrightarrow \tfrac{1}{2}(h + 1 + \sqrt{5h^2 - 2h + 1}) \leq k + h + 1 \\
&\Longleftrightarrow 5h^2 - 2h + 1 \leq (2k + h + 1)^2 \\
&\Longleftrightarrow 4h^2 \leq 4(k^2 + kh + h + k) \\
&\Leftarrow (k + 1 + \sqrt{5k^2 - 2k + 1})^2 \leq 4k^2 + \\
&\qquad 2(k + 1)(k - 1 + \sqrt{5k^2 - 2k + 1}) + 4k \\
&\Longleftrightarrow 4 \leq 4k
\end{aligned}
$$

which is true. Together these imply

$$h_2(k) = k + h(k) \qquad \text{or} \qquad k + h(k) - 1 .$$

We now show in the homogeneous case that this recurrence continues. Assume

$$h_{i-1} + h_i = h_{i+1} \quad \text{for} \quad i = 1, 2, \cdots, n \text{ where } n \geq 2.$$

We will prove that

$$h_n + h_{n+1} = h_{n+2} ,$$

that is

$$\tfrac{1}{2}(h_{n+1} - 1 + \sqrt{5h_{n+1}^2 - 2h_{n+1} + 1}) < h_n + h_{n+1} \le \tfrac{1}{2}(h_{n+1} + 1 + \sqrt{5h_{n+1}^2 - 2h_{n+1} + 1}).$$

The right side is equivalent to

$$(2h_n + h_{n+1} - 1)^2 \le 5h_{n+1}^2 - 2h_{n+1} + 1$$
$$\Leftrightarrow h_n^2 + h_n h_{n+1} - h_n \le h_{n+1}^2$$
$$\Leftrightarrow h_n^2 - h_n h_{n-1} - h_{n-1}^2 \le h_n$$
$$\Leftarrow |h_1^2 - h_1 h_0 - h_0^2| \le h_2 \le h_n$$
$$\Leftarrow \left|\tfrac{1}{4}(h_0 + 1 + \sqrt{5h_0^2 - 2h_0 + 1})^2 - \tfrac{1}{2}h_0(h_0 - 1 + \sqrt{5h^2 - 2h_0 + 1}) - h_0^2\right| \le h_2 \tag{3}$$
$$\Leftrightarrow \left|\tfrac{1}{2}(h_0 + 1 + \sqrt{5h_0^2 - 2h_0 + 1})\right| \le h_2$$
$$\Leftarrow \left|\tfrac{1}{2}(h_0 + 1 + \sqrt{5h_0^2 - 2h_0 + 1})\right| \le h_1 + 1 \le h_1 + h_0 = h_2 .$$

But this last statement is true, verifying the right side. The left side is equivalent to

$$5h_{n+1}^2 - 2h_{n+1} + 1 < (2h_n + h_{n+1} + 1)^2$$
$$\Leftrightarrow h_{n+1}^2 < h_n^2 + h_n h_{n+1} + h_{n+1} + h_n$$
$$\Leftrightarrow -(h_n^2 - h_n h_{n-1} - h_{n-1}^2) < 2h_n + h_{n-1} = h_{n+2}$$

which is certainly true in light of (3) and Theorem 6. Proof of the nonhomogeneous case uses essentially the same techniques, although it is more complicated and is therefore omitted.

It is natural to ask for which integers k the sequence $\{h_n(k)\}$ is homogeneous and for which it is nonhomogeneous.

Theorem 9. The sequence $\{h_n(k)\}$ is nonhomogeneous if and only if

(4) $$h^2(k) < k^2 - k + kh(k) .$$

Proof. Using Theorem 8 it follows that $\{h_n(k)\}$ is nonhomogeneous if and only if

$$k + h(k) = h_2(k) - 1$$

$$\iff \tfrac{1}{2}(h + 1 + \sqrt{5h^2 - 2h + 1}) < k + h$$

$$\iff 5h^2 - 2h + 1 < (2k + h - 1)^2$$

$$\iff h^2 < k^2 - k + kh .$$

However the characterization of the $k$ which obey (4) seems difficult. It appears that numbers of the form $k = F_m + 1$ $(m > 5)$ satisfy (4), but there are others.

From the recurrence relations of Theorem 8 we may establish the following generating functions using standard techniques. If $\{h_n(k)\}$ is homogeneous,

$$\frac{p(x)}{1 - x - x^2} = \sum_{n=0}^{\infty} h_n(k)\, x^n ,$$

where $p(x) = \{h(k) - k\}x + k$; if $\{h_n(k)\}$ is nonhomogeneous,

$$\frac{q(x)}{(1 - x)(1 - x - x^2)} = \sum_{n=0}^{\infty} h_n(k)\, x^n ,$$

where $q(x) = \{h_2(k) - 2h(k)\}x^2 + \{h(k) - 2k\}x + k.$

Finally we show an interweaving of the numbers $h_n(k)$ in Table 1.

Theorem 10. For $r \geq 0$ let $M_r(n)$ be the number of integers not greater than $n$ which do not appear in the sequence $\{h_r(j)\}_{j=1}^{\infty}$. Then for $n \geq r$,

$$M_r(h_n(k)) = h_n(k) - h_{n-r}(k) \ .$$

Proof. We begin by observing that if $s > t$ then $h(s) > h(t)$. First assume $n = r$, so that $h_{n-r}(k) = h_0(k) = k$. The $k$ distinct integers $h_r(1), \cdots, h_r(k)$ are the only members of $\{h_r(j)\}_{j=1}^{\infty}$ not greater than $h_r(k)$, so that $M_r(h_r(k)) = h_r(k) - k = h_r(k) - h_{r-r}(k)$, as required. Now assume $n \geq r$, and let $h_{n-r}(k) = m$. Then $h_n(k) = h_r(m)$, so by the above

$$M_r(h_r(m)) = h_r(m) - h_0(m)$$

which implies

$$M_r(h_n(k)) = h_n(k) - h_{n-r}(k) \ ,$$

the desired result.

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