Beginning with the golden rectangle with base 2 and altitude $\sqrt{5} - 1$, one may proceed to construct a sequence of numbers which represent altitudes (shortest sides) of the nested golden rectangles.

\[
\begin{align*}
\sqrt{5} - 1, & \quad 3 - \sqrt{5}, \quad 2\sqrt{5} - 4, \quad 7 - 3\sqrt{5}, \quad 5\sqrt{5} - 11, \quad 18 - 8\sqrt{5}, \quad \cdots
\end{align*}
\]

We shall call this the sequence of golden numbers. These numbers, as one may suspect, are closely related to Fibonacci numbers, as is suggested by Theorem 2 below. First, however, we need to observe that the $n$th golden number may be expressed by the following recursive formula:

\[
\text{Theorem 1. If } g_n \text{ denotes the } n\text{th golden number, then } g_n = \frac{1}{2} g_1 \cdot g_{n-1}.
\]

\[
\text{Proof. This follows immediately from the method of finding the altitude of a golden rectangle given its base (details left for the reader).}
\]

As an immediate consequence we have a corollary:

\[
g_n = \frac{(\sqrt{5} - 1)^n}{2^{n-1}}.
\]

We next observe after considering the first few golden numbers that

\[
\text{Theorem 2. } g_n = g_{n-2} - g_{n-1}
\]

\[
\text{Proof. Using the form for } g_n \text{ given in the Corollary to Theorem 1, we have}
\]

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Another rather interesting observation is that the coefficients of radical 5 appear to be the sequence of Fibonacci numbers with alternating signs. We may formalize the conjecture after observing that as a result of a multiplication by \((\sqrt{5} - 1)/2\), the signs of each term of the golden numbers alternate and the \(n\)th golden number may be expressed in the form

\[
g_n = (-1)^{n-1} a_n \cdot \sqrt{5} - b_n
\]

where \(a_n\) and \(b_n\) are positive integers.

**Theorem 3.** If

\[
g_n = (-1)^{n-1} [a_n \cdot \sqrt{5} - b_n]
\]

represents the \(n\)th golden number, then \(a_n\) is the \(n\)th Fibonacci number, \(F_n\).

**Proof.**

\[
g_{n+1} = g_n \cdot \frac{\sqrt{5} - 1}{2} = (-1)^{n-1} \left( \frac{5a_n - \sqrt{5}b_n - \sqrt{5}a_n + b_n}{2} \right)
\]

\[
= (-1)^n \left[ \frac{(a_n + b_n)}{2} \sqrt{5} - \frac{(5a_n + b_n)}{2} \right]
\]

\[
\therefore \ a_{n+1} = \frac{a_n + b_n}{2} \quad \text{and} \quad b_{n+1} = \frac{5a_n + b_n}{2}
\]

Then
IN THE SEQUENCE OF GOLDEN NUMBERS

\[
\begin{align*}
\frac{a_{n-1} + a_n}{2} &= \frac{a_{n-1} + b_{n-1}}{2} = \frac{3a_{n-1} + b_{n-1}}{2} \\
\end{align*}
\]

and

\[
\begin{align*}
\frac{a_n + b_n}{2} &= \frac{a_{n-1} + b_{n-1}}{2} + \frac{5a_{n-1} + b_{n-1}}{2} = \frac{3a_{n-1} + b_{n-1}}{2} \\
&= a_{n-1} + a_n \rightarrow a_n = F_n.
\end{align*}
\]

Yet another observation may be made from the sequence (1). It is stated:

**Theorem 4.** If \( g_n \) and \( g_{n+1} \) are any two successive golden numbers, then \( F_{n+1} \cdot g_n + F_n \cdot g_{n+1} = 2. \)

**Proof.** Using the representation for \( F_n \) developed in the proof of Theorem 3, we write

\[
F_{n+1} = \frac{F_n + b_n}{2} \rightarrow b_n = 2F_{n+1} - F_n
\]

Therefore, we may express \( g_n \) and \( g_{n+1} \) in terms of Fibonacci numbers only:

\[
g_n = (-1)^{n-1}(F_n \sqrt{5} + F_n - 2F_{n+1})
\]

and

\[
g_{n+1} = (-1)^n(F_{n+1} \sqrt{5} + F_{n+1} - 2F_{n+2}).
\]

Thus we obtain:

\[
F_{n+1} \cdot g_n + F_n \cdot g_{n+1} = (-1)^{n-1} \left[ F_n \cdot F_{n+1} \sqrt{5} + F_n \cdot F_{n+1} - 2F_{n+1}^2 \right]
\]

\[
- F_n \cdot F_{n+1} \sqrt{5} - F_n \cdot F_{n+1} + 2F_n \cdot F_{n+2}
\]
Recalling the fundamental identity

\[ F_{n-1} \cdot F_{n+1} = F_n^2 + (-1)^n, \quad n \geq 2, \]

It follows that

\[ F_n \cdot g_n + F_{n-1} \cdot g_{n+1} = (-1)^{n-1}[ -2F_n^2 + 2(F_{n+1}^2 + (-1)^n)] = 2 \]

Recalling the representation for \( g_n \) used in the proof of Theorem 4,

\[ g_n = (-1)^{n-1}(F_n \sqrt{5} + F_{n+1} - 2F_{n+1}) \]

we observe that

\[ F_n - 2F_{n+1} = F_n - 2[F_{n-1} + F_n] = -F_n - 2F_{n-1} \]

which gives us the following alternate forms for the \( n \)th golden number:

\[ g_n = (-1)^{n-1}(F_n \cdot g_1 - 2F_{n-1}) \]

or

\[ g_n = (-1)^{n-1}(\sqrt{5}F_n - L_n) \]

where \( L_n \) is the \( n \)th Lucas number. We now state our final result.

**Theorem 5.** \( g_n = (-1)^{n-1}(\sqrt{5}F_n - L_n) \)

**Proof.** Follows from the identity

\[ L_n = F_{n-1} + F_{n+1}. \]