ADVANCED PROBLEMS AND SOLUTIONS

Edited By

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Send all communications concerning Advanced Problems and Solutions to Raymond Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania, 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within three months after publication of the problems.

H-119 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Put

$$\overline{H}(m,n,p) = \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{k=0}^{p}(-1)^{i+j+k}\binom{i+j}{j}\binom{j+k}{k}\binom{k+m-i}{m-i}\binom{m-i+n-j}{n-j} \cdot \binom{n-j+p-k}{p-k}\binom{p-k+i}{i}$$

Show that $\overline{H}(m,n,p) = 0$ unless m, n, p are all even and that

$$\overline{H}(2m, 2n, 2p) = \sum_{r=0}^{\min(m,n,p)} (-1)^r \frac{(m+n+p-r)!}{r!\,r!\,(m-r)!\,(n-r)!\,(p-r)!} =$$

(The formula

$$\overline{H}(2m, 2n) = \binom{m+n}{m}^2 ,$$

where

$$\overline{H}(m,n) = \sum_{i=0}^{m}\sum_{j=0}^{n}(-1)^{i+j}\binom{i+j}{j}\binom{m-i+j}{j}\binom{i+n-j}{n-j}\binom{m-i+n-j}{n-j}$$

is proved in the Fibonacci Quarterly, Vol. 4 (1966), pp. 323-325.)

H-120 Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.

The Fibonacci polynomials are defined by

$$f_{n+1}(x) = x \cdot f_n(x) + f_{n-1}(x)$$
$$f_1(x) = 1, \quad f_2(x) = x .$$

If $z_r = f_r(x) \cdot f_r(y)$ then show that

(i) z_r satisfies the recurrence relation,

$$z_{n+4} - xy \cdot z_{n+3} - (x^2 + y^2 + 2)z_{n+2} - xy \cdot z_{n+1} + z_n = 0.$$

(ii) $(x + y)^2 \cdot \sum_1^n z_r = (z_{n+2} - z_{n-1}) - (xy - 1)(z_{n+1} - z_n).$

H-121 Proposed by H. H. Ferns, University of Victoria, Victoria, B.C., Canada.

Prove the following identity.

$$\sum_{i=1}^{n} \binom{n}{i} \left(\frac{F_k}{F_{m-k}}\right)^i F_{mi+\lambda} = \left(\frac{F_m}{F_{m-k}}\right)^n F_{nk+\lambda} - F_\lambda \quad (m \neq k) ,$$

where F_n is the n^{th} Fibonacci number, m, λ are any integers or zero and k is an even integer or zero.

Write the form the identity takes if k is an odd integer.

Find an analogous identity involving Lucas numbers.

H-122 Proposed by R. E. Whitney, Lock Haven State College, Lock Haven, Pa.

Let F_n denote the n^{th} Fibonacci number expressed in base 2. Consider the ordered array $F_1F_2F_3 \cdots$. Let g_n denote the n^{th} digit of this array. Find a formula for g_n. If possible, generalize for any base.

H-70 Proposed by C. A. Church, Jr., W. Virginia Univ., Morgantown, W. Virginia.

For $n = 2m$, show that the total number of k-combinations of the first n natural numbers such that no two elements i and $i + 2$ appear together in the same selection is F^2_{m+2} and if $n = 2m + 1$, the total is $F_{m+2}F_{m+3}$.

Solution and comments by the proposer.

For his quick solution of the "problème des ménages" Kaplansky [2] gives two results for combinations with restricted positions. We state them in the following form:

The number of k-combinations of the first n natural numbers, on a line, with no two consecutive is

$$\binom{n - k + 1}{k} \quad ; \quad 0 \le k \le \frac{n+1}{2}, \tag{1}$$

if arranged on a circle, so that 1 and n are consecutive, the number is

$$\frac{n}{n-k}\binom{n-k}{k}, \qquad 0 \le k \le \frac{n}{2}. \tag{2}$$

See also [4, p. 198]. Summed over k, (1) and (2) give the Fibonacci and Lucas numbers, respectively.

For the problem as stated we use (1).

The restriction that i and $i + 2$ cannot appear in any selection can be stated as (a) no two consecutive even integers appear and (b) no two consecutive odd integers appear.

If $n = 2m$, a k-combination with the stated restrictions will be made up of s integers from among the m even, no two consecutive, and $k - s$ from among the m odd, no two consecutive. Thus there are

$$\sum_{s=0}^{k}\binom{m - s + 1}{s}\binom{m - (k - s) + 1}{k - s} \tag{3}$$

k-combinations of the first 2m natural numbers such that i and $i + 2$ do not appear.

Summing (3) over k we get the total number

$$F^2_{m+2} = \sum_{k=0}^{2\left[\frac{m+1}{2}\right]} \sum_{s=0}^{k} \binom{m-s+1}{s}\binom{m-(k-s)+1}{k-s}$$

with the usual condition that

$$\binom{a}{b} = 0 \quad \text{for} \quad b > a \geq 0 .$$

For $n = 2m + 1$ we choose s from among the m even integers, no two consecutive, and $k - s$ from among the $m + 1$ odd integers, no two consecutive, to get that there are

$$\sum_{s=0}^{k} \binom{m-s+1}{s}\binom{m-(k-s)+2}{k-s} \tag{4}$$

k-combinations of the first $2m + 1$ natural numbers such that i and $i + 2$ do not appear.

Summed over k, (4) gives the total number

$$F_{m+2}F_{m+3} = \sum_{k=0}^{m+1} \sum_{s=0}^{k} \binom{m-s+1}{s}\binom{m-(k-s)+2}{k-s} .$$

It is also of interest to consider the circular analog of this problem by way of (2).

For $n = 2m$, 2 and 2m are taken to be consecutive as are 1 and $2m - 1$. By the same argument as before we find that there are

$$\sum_{s=0}^{k} \frac{m}{m-s}\binom{m-s}{s} \frac{m}{m-(k-s)}\binom{m-(k-s)}{k-s}$$

circular k-combinations such that i and $i + 2$ do not appear and a total of

$$L_m^2 = \sum_{k=0}^{2\left[\frac{m}{2}\right]} \sum_{s=0}^{k} \frac{m}{m-s}\binom{m-s}{s} \frac{m}{m-(k-s)}\binom{m-(k-s)}{k-s} .$$

For $n \doteq 2m+1$, 2 and $2m$ are consecutive as are 1 and $2m+1$ and we have the total

$$L_m L_{m+1} = \sum_{k=0}^{m} \sum_{s=0}^{k} \frac{m}{m-s}\binom{m-s}{s} \frac{m+1}{m-(k-s)+1}\binom{m-(k-s)+1}{k-s} .$$

Mixed results can also be obtained using both (1) and (2). For example, one can take linear combinations on the evens and circular combinations on the odds.

Remarks. The problem posed in H-70 first appeared in the literature in a paper by N. S. Mendelsohn [3]; an explicit formula was not obtained. The first explicit formula was given by M. Abramson [1, lemma 3]. Abramson's solution for the number of k-combinations such that i and i + 2 do not appear together is

$$\sum_{s=0}^{\left[\frac{k}{2}\right]} \binom{n-2k+s+2}{k-s}\binom{k-s}{s} .$$

REFERENCES

1. M. Abramson, "Explicit Expressions for a Class of Permutation Problems," Canad. Math. Bull., 7(1964), pp. 345-350.
2. I. Kaplansky, Solution of the "Problème des Ménages," Bull. Amer. Math Soc., 49(1943), 784-785.
3. N. S. Mendelsohn, "The Asymptotic Series for a Certain Class of Permutation Problems, Canad. J. of Math., 8(1956), pp. 234-244.
4. J. Riordan, An Introduction to Combinatorial Analysis, New York, 1958.

H-73 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let $f_0(x) = 0, \quad f_1(x) = 1$

and
$$f_{n+2}(x) = xf_{n+1}(x) + f_n(x), \qquad n \geq 0$$

and let $b_n(x)$ and $B_n(x)$ be the polynomials in H-69; show

$$f_{2n+2}(x) = xB_n(x^2),$$

and

$$f_{2n+1}(x) = b_n(x^2).$$

Thus there is an intimate relationship between the Fibonacci polynomials, $f_n(x)$ and the Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$.

Solution by Douglas Lind, Univ. of Virginia, Charlottesville, Virginia.

Using the explicit representations of $B_n(x)$ and $b_n(x)$ given in H-69, and of $f_n(x)$ given in B-74, we find

$$xB_n(x^2) = \sum_{r=0}^{n} \binom{n+r+1}{n-r} x^{2r+1} = \sum_{r=0}^{n} \binom{2n-r+1}{r} x^{2n-2r+1} = f_{2n+2}(x),$$

$$b_n(x^2) = \sum_{r=0}^{n} \binom{n+r}{n-r} x^{2r} = \sum_{r=0}^{n} \binom{2n-r}{r} x^{2n-2r} = f_{2n+1}(x).$$

These relations have been given by R. A. Hayes ["Fibonacci and Lucas Polynomials," (Master's Thesis); equations (3.4-1) and (3.4-2)].

H-77 Proposed by V.E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Show

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{2k+2j+1} = 5^n L_{2n+2k+2}$$

for all integers k. Set $k = -(n+1)$ and derive

$$\sum_{j=0}^{n} \binom{2n+1}{n-j} F_{2j+1} = 5^n,$$

a result of S. G. Guba Problem No. 174, Issue No. 4, July-August 1965, p. 73 of Matematika V. Skŏle.

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Since

$$L_n = \alpha^n + \beta^n, \quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} ,$$

where

$$\alpha = \frac{1}{2}(1 + \sqrt{5}), \quad \beta = \frac{1}{2}(1 - \sqrt{5}) ,$$

$$1 + \alpha^2 = \alpha\sqrt{5} , \quad 1 + \beta^2 = -\beta\sqrt{5} ,$$

it follows that

$$\begin{aligned}
\sum_{j=0}^{n} \binom{n}{j} F_{k+2j} &= \sum_{j=0}^{n} \binom{n}{j} \frac{\alpha^{k+2j} - \beta^{k+2j}}{\alpha - \beta} \\
&= \frac{\alpha^k (1 + \alpha^2)^n - \beta^k (1 + \beta^2)^n}{\alpha - \beta} \\
&= \frac{(\alpha^{k+n} - (-1)^n \beta^{k+n})(\sqrt{5})^n}{\sqrt{5}} \\
&= \begin{cases} 5^{(n-1)/2} L_{k+n} & (n \text{ odd}) \\ 5^{n/2} F_{k+n} & (n \text{ even}) , \end{cases}
\end{aligned}$$

thus generalizing the stated result. In particular, for $k = -n$, we get

$$\sum_{j=0}^{n} \binom{n}{j} F_{-n+2j} = \begin{cases} 2 \cdot 5^{(n-1)/2} & (n \text{ odd}) \\ 0 & (n \text{ even}) . \end{cases}$$

Note that

$$\sum_{j=0}^{2n+1} \binom{2n+1}{j} F_{-2n-1+2j} = \sum_{j=0}^{n} \binom{2n+1}{j} F_{-2n-1+2j} + \sum_{j=n+1}^{2n+1} \binom{2n+1}{j} F_{-2n-1+2j}$$

$$= \sum_{j=0}^{n} \binom{2n+1}{j} F_{2n+1-2j} + \sum_{j=0}^{n} \binom{2n+1}{j} F_{2n+1-2j}$$

$$= 2 \sum_{j=0}^{n} \binom{2n+1}{n-j} F_{2j+1} \quad ,$$

so that

$$\sum_{j=0}^{n} \binom{2n+1}{n-j} F_{2j+1} = 5^n \quad .$$

Similarly, we have

$$\sum_{j=0}^{n} \binom{n}{j} L_{k+2j} = \alpha^k(1+\alpha^2)^n + \beta^k(1+\beta^2)^n$$

$$= (\alpha^{k+n} + (-1)^n \beta^{k+n})(\sqrt{5})^n$$

$$= \begin{cases} 5^{n/2} L_{k+n} & (n \text{ even}) \\ 5^{(n+1)/2} F_{k+n} & (n \text{ odd}) \quad . \end{cases}$$

In particular, since

$$\sum_{j=0}^{2n} \binom{2n}{j} L_{-2n+2j} = -2\binom{2n}{n} + \sum_{j=0}^{n} \binom{2n}{j} L_{-2n+2j} = \sum_{n}^{2n} \binom{2n}{j} L_{-2n+2j}$$

$$= -2\binom{2n}{n} + 2\sum_{j=0}^{n} \binom{2n}{n-j} L_{2j} \quad ,$$

so that

$$\sum_{j=0}^{n} \binom{2n}{n-j} L_{2j} = \binom{2n}{n} + 5^n \quad .$$

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