

FIBONACCI NUMBERS AND SOME PRIME RECIPROCAL

R. S. BUCKNELL
Chiangmai, Thailand

The series of Fibonacci numbers has been shown to bear some interesting relationships to the reciprocals of certain prime numbers. For instance, Maxey Brooke and C. R. Wall set up as Problems B-14 (Fibonacci Quarterly, Vol. 1 (1963), No. 2, p. 86) to show

$$(1) \quad \sum_{n=1}^{\infty} F_n 10^{-n} = 10/89 ,$$

$$(2) \quad \sum_{n=1}^{\infty} (-1)^{n+1} F_n 10^{-n} = 10/109 ,$$

where $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$.

The relationship (1) is but a special case of the general property

$$(3) \quad \sum_{n=1}^{\infty} F_n x^{-n-1} = \frac{1}{x^2 - x - 1}$$

where x , an integer >2 , is the radix in terms of which the number $(x^2 - x - 1)$ and the Fibonacci numbers are expressed. Equation (3) is readily proved by considering

$$\begin{aligned} (x^2 - x - 1) \sum_{n=1}^{\infty} F_n x^{-n-1} &= x^2 \sum_{n=1}^{\infty} F_n x^{-n-1} - x \sum_{n=1}^{\infty} F_n x^{-n-1} - \sum_{n=1}^{\infty} F_n x^{-n-1} \\ &= \sum_{n=1}^{\infty} F_n x^{-n+1} - \sum_{n=1}^{\infty} F_n x^{-n} - \sum_{n=1}^{\infty} F_n x^{-n-1} \\ &= \left[\sum_{n=1}^{\infty} F_{n+2} x^{-n-1} + F_1 + F_2 x^{-1} \right] - \left[\sum_{n=1}^{\infty} F_{n+1} x^{-n-1} + F_1 x^{-1} \right] \end{aligned}$$

$$- \left[\sum_{n=1}^{\infty} F_n x^{-n-1} \right] = 1$$

since

$$F_1 = F_2 = 1$$

and

$$\sum_{n=1}^{\infty} F_{n+2} x^{-n-1} - \sum_{n=1}^{\infty} F_{n+1} x^{-n-1} - \sum_{n=1}^{\infty} F_n x^{-n-1} = 0 .$$

Note that $x^2 - x - 1$ may be composite (e. g., for $x = 8, 13$), but that $x^2 - x - 1$ and x are relatively prime.

Another interesting relationship that has been discovered to exist between the series of Fibonacci numbers and the number $1/N = 1/(x^2 - x - 1)$ is exemplified by the special case where $N = 109$. It is found that the 108-digit period of $1/109$ is

0091743119266055045871559633027522935779816513761467889909256880733944-
95412844036697247706422018348623853211 .

A generalization is possible for the number $N = x^2 - x - 1 - y^2 + y - 1$ (x an integer >2 , $y = x - 1$) when $1/N$ is expanded in terms of radix y . It will be shown that if the period of $1/N$ is the $(N - 1)$ -digit number

$$P = \frac{y^{N-1} - 1}{N} ,$$

then the number

$$\sum_{n=1}^{N-1} F_n y^{n-1}$$

has as its last $N - 1$ digits the number P .

Let the residue, R , be defined by

$$(4) \quad R = \sum_{n=1}^{N-1} F_n y^{n-1} - P .$$

The expression

$$\sum_{n=1}^{N-1} F_n y^{n-1}$$

will be summed using

$$F_n = \frac{a^n - b^n}{a - b} ,$$

where

$$a = \frac{1 + \sqrt{5}}{2} , \quad b = \frac{1 - \sqrt{5}}{2} .$$

Then

$$\begin{aligned} \sum_{n=1}^{N-1} F_n y^{n-1} &= \sum_{n=1}^{N-1} \left[\left(\frac{a^n - b^n}{a - b} \right) \cdot y^{n-1} \right] \\ &= \sum_{n=1}^{N-1} [(ay)^n - (by)^n] / (a - b)y \\ &= \left[\frac{1 - (ay)^N}{1 - ay} - \frac{1 - (by)^N}{1 - by} \right] / (a - b)y \\ &= \frac{y^N F_{N-1} + y^{N-1} F_N - 1}{y^2 + y - 1} - \frac{y^{N-1} - 1}{y^2 + y - 1} \\ &= \frac{y^{N-1}}{y^2 + y - 1} [y F_{N-1} + F_N - 1] . \end{aligned}$$

Now consider the term

$$\frac{y^{N-1}}{y^2 + y - 1}$$

Clearly no factor of y will divide $y^2 + y - 1$, hence y^{N-1} and $y^2 + y - 1$ are relatively prime, and since R is an integer, y^{N-1} divides R . Thus R is a number ending in $N - 1$ zeros when expressed in terms of radix y . Since P contains not more than $N - 1$ digits, it follows that the number

$$\sum_{n=1}^{N-1} F_n y^{n-1} = P + R$$

has as its last $(N - 1)$ digits the number P .

NOTICE TO ALL SUBSCRIBERS!!!

Please notify the Managing Editor AT ONCE of any address change. The Post Office Department, rather than forwarding magazines mailed third class, sends them directly to the dead-letter office. Unless the addressee specifically requests the Fibonacci Quarterly to be forwarded at first class rates to the new address, he will not receive it. (This will usually cost about 30 cents for first-class postage.) If possible, please notify us AT LEAST THREE WEEKS PRIOR to publication dates: February 15, April 15, October 15, and December 15.

All subscription correspondence should be addressed to Brother A. Brousseau, St. Mary's College, Calif. All checks (\$4.00 per year) should be made out to the Fibonacci Association or the Fibonacci Quarterly. Manuscripts intended for publication in the Quarterly should be sent to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, Calif. All manuscripts should be typed, double-spaced. Drawings should be made the same size as they will appear in the Quarterly, and should be done in India ink on either vellum or bond paper. Authors should keep a copy of the manuscripts sent to the editors.
