EXISTENCE OF ARBITRARILY LONG SEQUENCES OF CONSECUTIVE MEMBERS IN ARITHMETIC PROGRESSIONS DIVISIBLE BY ARBITRARILY MANY DIFFERENT PRIMES

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It is well known that there exist arbitrarily long sequences of consecutive positive integers that are all composite, e.g., $(n + 1)! + 2$, $(n + 1)! + 3, \cdots$, $(n + 1)! + (n + 1)$. This statement can also be formulated thus: for any given positive integer n there exist n consecutive composite positive integers each of which has at least one prime divisor. The following is a twofold generalization of the last statement.

Theorem. In any infinite arithmetic progression

(1) $\qquad ax + b, \quad a, b \text{ integers}, \quad a \neq 0, \quad x = 1, 2, 3, \cdots$

and for any two positive integers, n, ν, there exist n consecutive members each of which is divisible by at least ν different primes.

Proof. (By induction on ν). Since $a \neq 0$, we have $a < 1$ or $a \lesssim 1$. We may suppose, without loss of generality, $a \gtrsim 1$, since if $a < 1$ we can consider the progression $-ax - b$, the members of which have the same absolute values as the corresponding members of (1). Thus for $x > (1 - b)/a$, (1) is an increasing sequence of positive integers >1. Since any integer >1 is divisible by at least one prime, our statement is valid for $\nu = 1$. From the validity of the statement for ν we shall prove its validity for $\nu + 1$. As a matter of fact, let $2 \leq a_1 < a_2 < \cdots < a_n$ be n consecutive members of (1) each of which is divisible by at least ν different primes. Consider the sequence of n consecutive positive integers $(a_n)!^2a+a_1, (a_n)!^2a+a_2, \cdots, (a_n)!^2a+a_n$. For $2 \leq a_1 \leq a_k \leq a_n$ we have

$$(a_n)!^2a + a_k = a_k\left[\frac{(a_n)!^2a}{a_k} + 1\right] = a_k\left[\frac{(2 \cdot 3 \cdot 4 \cdots a_{k-1} \cdot a_k \cdot a_{k+1} \cdots a_n)(a_n)!\,a}{a_k} + 1\right]$$

$$= a_k\left[(2 \cdot 2 \cdot 4 \ldots a_{k-1} \cdot a_{k+1} \cdots a_n)(2 \cdot 3 \cdot 4 \cdots a_k \cdots a_n)a +\right.$$

The sum in brackets is composed of two terms, one divisible by a_k, the other being 1. Thus, this sum is coprime with a_k, and since it is greater than 1, it is divisible by a prime not dividing a_k. Hence $(a_n)!^2a + a_k$ is divisible by $\nu + 1$ different primes, for any $1 \leq k \leq n$. On the other hand, since a_k is a member of (1), thus of the form $ax + b$, we have $(a_n)!^2a + a_k \equiv b \pmod{a}$, thus $(a_n)!^2a + a_k$ is a member of (1), which completes the proof of the theorem.

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