A PROPERTY OF LINEAR RECURSION RELATIONS

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If one selects the basic Fibonacci recursion relation,

$$(1) \qquad U_{n+2} - U_{n+1} - U_n = 0; \ (n \geq 0) \ U_0 = 0, \ U_1 = 1 ,$$

and applies the well-known series transformation [1],

$$(2) \qquad y(t) = \sum_0^{\infty} U_k t^k / k! ,$$

one obtains a linear differential equation with a characteristic equation,

$$(3) \qquad V_n \equiv n^2 - n - 1 = 0 .$$

It is easily verified that the recursion relation satisfied by $\{V_n\}$ is

$$(4) \qquad V_{n+2} - 2V_{n+1} + V_n = 2 .$$

It seemed reasonable to consider the relationship between (1) and (4). When the relationship was investigated, a rather unusual result was obtained. A characterization of recursion relations of polynomials was also obtained from the results. To carry out the investigation, it was expedient to introduce some terminology.

Let a linear recursion relation of order p with constant coefficients be denoted by

$$L_1(U) \equiv \sum_{i=n}^{n+p} a_{i-n} U_i = b \text{ for all } n \geq 1; \ a_p = 1 .$$

When we apply the series transform (2) to the above, we obtain

$$\sum_{0}^{p} a_i y^{(i)} = be^t \quad .$$

The characteristic equation of this differential equation is,

$$\sum_{0}^{p} a_i n^i = 0 \quad .$$

Let

$$V_n \equiv \sum_{0}^{p} a_i n^i; \quad (n \geq 1) \quad V_0 = a_0 \quad .$$

Define

$$L_2(V) = \sum_{i=n}^{n+p} d_{i-n} V_i = C; \quad d_p = 1 \quad ,$$

as the conjugate recursion relation of $L_1(U)$. The d's and C will shortly be determined explicitly. Since $d_p = 1$, it may be shown, as follows, that for fixed order p, $L_2(V)$ is unique.

Clearly for any particular recurrence relation, $L_1(U) = b$, $\{V_n\}$ is unique. Suppose $L_2(V)$ were not unique. Then

$$(5) \qquad L_2'(V) = \sum_{i=n}^{n+p} d'_{i-n} V_i = C'$$

and

$$(6) \qquad L_2''(V) = \sum_{i=n}^{n+p} d''_{i-n} V_i = C''; \quad d'_p = d''_p = 1 \quad ,$$

are two distinct normalized recurrence relations for $\{V_n\}$. For any fixed p, the series transformed differential equations for (5) and (6), effected by (2), would be distinct p^{th} order linear differential equations with identical boundary conditions and the same solution,

$$y(t) = \sum_{0}^{\infty} V_i t^i / i! \quad .$$

Note that the series for y(t) converges for all t, since $|V_n|$ is dominated by

$$\operatorname*{Max}_{i=0,\cdots,p} |a_i| \sum_{0}^{p} n^i$$

and thus $|V_n| = O(n^p)$.

By linearity properties of the solutions of linear differential equations with constant coefficients, this is impossible. Hence (5) and (6) are identical.

Thus $L_2(V)$ is the normalized recursion relation of order p, satisfied by $\{V_n\}$. We shall say that $L_1(U)$ is self-conjugate if and only if

$$L_1(U) = L_2(U) \quad .$$

Before we state and prove the central theorem, we shall need a lemma and two preliminary theorems.

<u>Lemma:</u>

$$\sum_{j=0}^{p} (-1)^j (n + p - j)^{p-k} \binom{p}{j} = \begin{cases} 0 & \text{if } k = 1, 2, \cdots, p \\ p! & \text{if } k = 0 \ . \end{cases}$$

The proof of the above is an elementary albeit tedious exercise in induction and is given as elementary problem, E1253, in [2].

Theorem 1:

$$V_n = \sum_0^p a_i n^i \; ; \quad a_p = 1$$

implies

$$J \equiv \sum_{j=0}^{p} (-1)^j \binom{p}{j} V_{n+p-j} = p!$$

Proof. If we use the polynomial expression for V_n, in J, the coefficient of a_{p-k} in J is

$$\sum_{j=0}^{p} (-1)^j (n + p - j)^{p-k} \binom{p}{j}$$

By the lemma,

$$J = p!\, a_p = p! \qquad \text{QED.}$$

Theorem 2:

$$L_1(U) = b; \quad a_p = 1$$

implies

$$L_2(V) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} V_{n+p-j} = p!$$

Proof. The characteristic equation of the series transform of $L_1(U)$ is

$$\sum_{0}^{p} a_i n^i = 0 .$$

Thus

$$V_n = \sum_{0}^{p} a_i n^i \qquad \text{QED.}$$

and by Theorem 1, the result follows.

Theorem 3: If $L_1(U)$ is of order p, then a necessary and sufficient condition that $L_1(U)$ be self-conjugate is that

$$L_1(U) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} V_{n+p-j} = p!$$

The proof follows from Theorem 2 and the uniqueness of $L_2(V)$.

In the light of the above we have

Corollary: Every polynomial of degree p has the same recursion relation and a recursion relation of the type given in Theorem 3 yields a polynomial expression in closed form.

If we choose $p = 2$, $a_1 = a_0 = -1$, $b = 0$ we obtain (1) above. The conjugate of (1) is (4). Thus the Fibonacci relation is not self-conjugate.

It would be interesting to see if there are other classes of functions which yield a fixed recursion relation for all members of some subclasses of the class of functions. Since the types of solutions of linear recursion formulae with constant coefficients are quite restricted, one would have to consider more general relations to obtain results of much consequence.

REFERENCES

1. James A. Jeske, "Linear Recurrence Relations — Part I," Fibonacci Quarterly, Vol. 1, No. 2, April, 1963, pp. 69-74.
2. American Math. Monthly, Vol. 64, No. 8, October, 1957, p. 594.

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