FIBONACCI NUMBERS AND SOME PRIME RECIPROCALS

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The series of Fibonacci numbers has been shown to bear some interesting relationships to the reciprocals of certain prime numbers. For instance, Maxey Brooke and C. R. Wall set up as Problems B-14 (Fibonacci Quarterly, Vol. 1 (1963), No. 2, p. 86) to show

(1) $$\sum_{n=1}^{\infty} F_n 10^{-n} = 10/89 ,$$

(2) $$\sum_{n=1}^{\infty} (-1)^{n+1} F_n 10^{-n} = 10/109 ,$$

where $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$.

The relationship (1) is but a special case of the general property

(3) $$\sum_{n=1}^{\infty} F_n x^{-n-1} = \frac{1}{x^2 - x - 1}$$

where x, an integer >2, is the radix in terms of which the number $(x^2 - x - 1)$ and the Fibonacci numbers are expressed. Equation (3) is readily proved by considering

$$(x^2 - x - 1) \sum_{n=1}^{\infty} F_n x^{-n-1} = x^2 \sum_{n=1}^{\infty} F_n x^{-n-1} - x \sum_{n=1}^{\infty} F_n x^{-n-1} - \sum_{n=1}^{\infty} F_n x^{-n-1}$$

$$= \sum_{n=1}^{\infty} F_n x^{-n+1} - \sum_{n=1}^{\infty} F_n x^{-n} - \sum_{n=1}^{\infty} F_n x^{-n-1}$$

$$= \left[\sum_{n=1}^{\infty} F_{n+2} x^{-n-1} + F_1 + F_2 x^{-1}\right] - \left[\sum_{n=1}^{\infty} F_{n+1} x^{-n-1} + F_1 x^{-1}\right]$$

$$- \left[\sum_{n=1}^{\infty} F_n x^{-n-1} \right] = 1$$

since

$$F_1 = F_2 = 1$$

and

$$\sum_{n=1}^{\infty} F_{n+2} x^{-n-1} - \sum_{n=1}^{\infty} F_{n+1} x^{-n-1} - \sum_{n=1}^{\infty} F_n x^{-n-1} = 0 .$$

Note that $x^2 - x - 1$ may be composite (e. g., for $x = 8, 13$), but that $x^2 - x - 1$ and x are relatively prime.

Another interesting relationship that has been discovered to exist between the series of Fibonacci numbers and the number $1/N = 1/(x^2 - x - 1)$ is exemplified by the special case where $N = 109$. It is found that the 108-digit period of 1/109 is

0091743119266055045871559633027522935779816513761467889909256880733944-
95412844036697247706422018348623853211 .

A generalization is possible for the number $N - x^2 - x - 1 - y^2 + y - 1$ (x an integer >2, $y = x - 1$) when $1/N$ is expanded in terms of radix y. It will be shown that if the period of $1/N$ is the $(N - 1)$-digit number

$$P = \frac{y^{N-1} - 1}{N} ,$$

then the number

$$\sum_{n=1}^{N-1} F_n y^{n-1}$$

has as its last $N - 1$ digits the number P.

Let the residue, R, be defined by

$$R = \sum_{n=1}^{N-1} F_n y^{n-1} - P . \tag{4}$$

The expression

$$\sum_{n=1}^{N-1} F_n y^{n-1}$$

will be summed using

$$F_n = \frac{a^n - b^n}{a - b} ,$$

where

$$a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2} .$$

Then

$$\begin{aligned}
\sum_{n=1}^{N-1} F_n y^{n-1} &= \sum_{n=1}^{N-1} \left[\left(\frac{a^n - b^n}{a - b} \right) \cdot \, y^{n-1} \right] \\
&= \sum_{n=1}^{N-1} \left[(ay)^n - (by)^n \right] / (a - b)y \\
&= \left[\frac{1 - (ay)^N}{1 - ay} - \frac{1 - (by)^N}{1 - by} \right] \Big/ (a - b)y \\
&= \frac{y^N F_{N-1} + y^{N-1} F_N - 1}{y^2 + y - 1} - \frac{y^{N-1} - 1}{y^2 + y - 1} \\
&= \frac{y^{N-1}}{y^2 + y - 1} \left[y F_{N-1} + F_N - 1 \right] .
\end{aligned}$$

Now consider the term

$$\frac{y^{N-1}}{y^2 + y - 1}$$

Clearly no factor of y will divide $y^2 + y - 1$, hence y^{N-1} and $y^2 + y - 1$ are relatively prime, and since R is an integer, y^{N-1} divides R. Thus R is a number ending in N - 1 zeros when expressed in terms of radix y. Since P contains not more than N - 1 digits, it follows that the number

$$\sum_{n=1}^{N-1} F_n y^{n-1} = P + R$$

has as its last (N - 1) digits the number P.

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