REMARKS ON TWO RELATED SEQUENCES OF NUMBERS

DANIEL C. FIELDER
Georgia Institute of Technology, Atlanta, Georgia

1. INTRODUCTION

The expansion of $(x+y)^n$ usually takes the form

$$(x+y)^n = \sum_{k=1}^{n+1} C(n,k)x^{n-k+1}y^{k-1}, \tag{1}$$

where $C(n,k)$ are the well-known binomial coefficients and are sequences of integers generated by the expansion (1). Another device for obtaining the $C(n,k)$ is, of course, Pascal's triangle.

Different sequences of numbers can be obtained from the coefficients resulting from the expansion of $(x+y)^n$ in terms of $(x^k+y^k)(xy)^{n-k}$. Further, a sort of inverse can be obtained by expressing (x^n+y^n) in terms of $(x+y)^k(xy)^{n-k}$. In both cases the coefficients share characteristics with certain binomial coefficients and terms from sums of powers of roots of selected polynomials. In the inverse sequences, except for appropriate changes in sign, the numerical coefficients are those observed in a recently proposed approach to the generation of Lucas numbers from partitions of numbers [1]. The relationship between partitions of numbers and both sequence is outlined briefly.

2. SEQUENCE OF THE FIRST KIND

For brevity, let $(x+y) = u = u_1$ (interchangeably), let $(x^k+y^k) = u_k$, and let $(xy) = v$. The numerical coefficients of the resultant direct expansion shown below are called coefficients of the first kind.

$$\begin{aligned}
u &= u_1 \\
u^2 &= u_2 + 2v^2 \\
u^3 &= u_3 + 3u_1v^2 \\
u^4 &= u_4 + 4u_2v^2 + 6v^4 \\
u^5 &= u_5 + 5u_3v^2 + 10u_1v^4 \\
u^6 &= u_6 + 6u_4v^2 + 15u_2v^4 + 20v^6 \\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{2}$$

As might be suspected, the coefficients are binomial coefficients without the symmetrically repeated coefficients of the expansion (1). The coefficients of (2) form the half-Pascal triangle enclosed by solid lines below.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}$$

3. SEQUENCE OF THE SECOND KIND

A rearrangement of (2) yields

$$\begin{aligned}
u_1 &= u \\
u_2 &= u^2 - 2v^2 \\
u_3 &= u^3 - 3uv^2 \\
u_4 &= u^4 - 4u^2v^2 + 2v^4 \\
u_5 &= u^5 - 5u^3v^2 + 5uv^4 \\
u_6 &= u^6 - 6u^4v^2 + 9u^2v^4 - 2v^6 \\
&\cdots\cdots\cdots\cdots
\end{aligned} \tag{3}$$

If the minus signs are temporarily neglected in (3), the diagram below illustrates one of the simple additive methods by which the coefficients can be obtained.

$$\begin{array}{cccccl}
1 & + & & & & 1 + 2 = 3 \\
1 & 2 & & & & 2 + 0 = 2 \\
1 & 3 & 0 & & & \\
1 & 4 & 2 & 0 & & 4 + 5 = 9 \\
1 & 5 & 5 & 0 & & \\
1 & 6 & 9 & 2 & &
\end{array} \tag{4}$$

If signs are neglected, it is interesting to note that the sum of the coefficients for any given index is identically the Lucas number of that index. Additional comments on this will be made later.

4. INTERRELATIONS

In [1], the sums of the powers of roots of

$$f(x) = x^n - x^{n-1} - x^{n-2} - \cdots - 1 \tag{5}$$

were obtained from a previously developed tabulation of Newton's formulas for powers of roots. The first few entries of that tabulation are given below without literal coefficients and without negative signs.

$$
\begin{aligned}
S_1 &= 1, \\
S_1 &= 1 + 2, \\
S_3 &= 1 + 3 + 3, \\
S_4 &= 1 + 4 + (4 + 2) + 4, \\
S_5 &= 1 + 5 + (5 + 5) + (5 + 5) + 5, \\
S_6 &= 1 + 6 + (6 + 9) + (6 + 12 + 2)(6 + 6 + 3) + 6, \\
S_7 &= 1 + 7 + (7 + 14) + (7 + 21 + 7) + (7 + 14 + 7 + 7) + (7 + 7 + 7) + 7,
\end{aligned}
\tag{6}
$$

Except for a missing final 1, the numbers as grouped in (6) are complete sets of binomial coefficients; hence, by selecting the appropriate numbers from (6), the coefficients for the first kind of sequence are readily obtained.

The extraction of the coefficients for the sequence of the second kind is more interesting. The same sets of numbers as those for the first kind of coefficients are considered. If in (6) a number is not parenthesized, it is one of the second kind coefficients as well as one of the first kind coefficients. However, it can be observed that whereas a first kind coefficient is equal to the sum of numbers within parentheses, the corresponding second kind coefficient is equal to the last number only of the numbers included within parentheses.

Without repeating the details covered in [1], it can be stated that the second kind coefficients be used to obtain the powers of roots of (5) for the case $n = 2$. Proper choice of sign leads to the ultimate identification.

In the previous paper [1], it was shown that the k^{th} Lucas number can be generated from the two-part partitions of k. The sum of the terms resulting from operations on the partitions is equal to the k^{th} Lucas number. The same operation on partitions can be used for finding the second kind coefficients.

However, here the individual terms, not the sum, are used. Proper choice of sign must be made since the partition method generates only positive numbers. It may be added that this latter method is of advantage only if a rapid and convenient means for obtaining partitions is available.

REFERENCES

1. D. C. Fielder, "Certain Lucas-Like Sequences and their Generation by Partitions of Numbers," Fibonacci Quarterly, Vol. 5 , No. 4 , pp. 319-324.

* * * * *

RECURRING SEQUENCES

Review of Book by Dov Jarden
By Brother Alfred Brousseau

For some time the volume, Recurring Sequences, by Dov Jarden has been unavailable, but now a printing has been made of a revised version. The new book contains articles published by the author on Fibonacci numbers and related matters in Riveon Lematematika and other publications. A number of these articles were originally in Hebrew and hence unavailable to the general reading public. This volume now enables the reader to become acquainted with this extensive material (some thirty articles) in convenient form.

In addition, there is a list of Fibonacci and Lucas numbers as well as their known factorizations up to the 385th number in each case. Many new results in this section are the work of John Brillhart of the University of San Francisco and the University of California.

There is likewise, a Fibonacci bibliography which has been extended to include articles to the year 1962.

This valuable reference for Fibonacci fanciers is now available through the Fibonacci Association for the price of $6.00. All requests for the volume should be sent to Brother Alfred Brousseau, Managing Editor, St. Mary's College, Calif., 94575.

* * * * *

* * * * *