A GENERATING FUNCTION ASSOCIATED WITH THE GENERALIZED STIRLING NUMBERS

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1. INTRODUCTION

E. T. Bell [2] has defined a set of generalized Stirling numbers of the second kind $S_k(n,r)$; the numbers $S_1(n,r)$ are the ordinary Stirling numbers of the second kind. Letting $\lambda(n)$ denote the number of odd $S_1(n + 1, 2r + 1)$ Carlitz [3] has shown that

$$\sum_{n=0}^{\infty} \lambda(n)x^n = \prod_{n=0}^{\infty} (1 + x^{2^n} + x^{2^{n+1}}) .$$

In Section 3, we shall determine the generating function for the number of odd generalized Stirling numbers $S_2(n,r)$. Indeed we shall prove the following theorem.

Theorem. Let $\omega(n)$ denote the number of odd generalized Stirling numbers $S_2(n + r, 4r)$; then

$$\sum_{n=0}^{\infty} \omega(n)x^n = \prod_{n=0}^{\infty} (1 + x^{3\cdot 2^n} + x^{2^{n+2}}) .$$

Later Carlitz [4] obtained the generating function for the number of $S_1(n, r)$ that are relatively prime to p for any given prime p. It would be of interest to obtain such a generating function for the generalized Stirling numbers $S_k(n, r)$. At present the apparent difficulty with the method used herein is that, except for the case $k = 2$ and $p = 2$, the basic recurrence (2.4) for $S_k(n, r)$ with $k > 1$ is a recurrence of more than three terms, whereas for the cases that have been solved we had a three-term recurrence. In Section 4, we shall discuss this problem for the numbers $S_2(n, r)$ and the prime $p = 3$; several congruences will also be obtained for this case.

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2. PRELIMINARIES

The numbers $S_k(n, r)$ may be defined by introducing an operator τ which transforms t^n into $(e^t - 1)^n$. Powers of τ are defined recursively as follows:

$$\tau^u t^n = \tau(\tau^{u-1}t^n) , \tag{2.1}$$

where u is a positive integer. We shall also define $\tau^0 t^n = t^n$. The generalized Stirling numbers are then defined by

$$\tau^k t^r = r! \sum_{n=0}^{\infty} S_k(n, r) \frac{t^n}{n!} . \tag{2.2}$$

Hence $S_1(n, r)$ is the ordinary Stirling number of the second kind (see [5, pp. 42-43]) and $S_0(n, r) = \delta(n, r)$, the Kronecker delta. From (2.1) and (2.2) we can readily see [2, p. 93] that

$$S_{j+k}(n, r) = \sum_{i=r}^{n} S_j(n, i) S_k(i, r) . \tag{2.3}$$

Hence the numbers $S_k(n, r)$ can be derived from the ordinary Stirling numbers of the second kind by repeated matrix multiplication (see [5, p. 34]).

Becker and Riordan [1] have studied some of the arithmetic properties of these numbers; in particular, they obtained for $S_k(n, r)$ the period modulo p, a prime. In the same paper they derived the following basic recurrence modulo p (equation (5.4)):

$$\begin{aligned} S_k(n + p^s, r) \equiv & \sum_{j=0}^{k-1} \sum_i \binom{s + j - 1}{j} S_j(n, i) S_{k-j}(i + 1, r) \\ & + \sum_{j=1}^{s} \binom{s + k - 1 - j}{k - 1} S_k(n, r - p^j) \pmod{p} . \end{aligned} \tag{2.4}$$

3. PROOF OF THEOREM

For $p = 2$ we have from (2.4) that

$$S_2(n + 4, r) \equiv S_2(n + 1, r) + S_2(n, r - 4) \pmod 2$$

Hence if we let

$$S_n(x) = \sum_{r=0}^{n} S_2(n, r)x^r , \tag{3.1}$$

it follows that

$$S_{n+4}(x) + S_{n+1}(x) + x^4 S_n(x) \equiv 0 \pmod 2 . \tag{3.2}$$

Let α_1, α_2, α_3, and α_4 be the roots of the equation

$$y^4 + y + x^4 = 0$$

in $F[y]$, where $F = GF(2, x)$, the function field obtained by adjoining the indeterminate x to the finite field $GF(2)$. Also let

$$\phi_n(x) = \sum_{j=1}^{4} \alpha_j^n . \tag{3.3}$$

Then from the definition of the α's we see that

$$\phi_0(x) = \phi_1(x) = \phi_2(x) = \phi_4(x) = 0, \ \phi_3(x) = 1 .$$

Moreover

$$\phi_{n+4}(x) = \phi_{n+1}(x) + x^4\phi_n(x) ; \tag{3.4}$$

hence

$$\phi_5(x) = 0, \quad \phi_6(x) = 1 .$$

Now put

$$\overline{S}_n(x) = (x^3 + x + 1)\phi_n(x) + x^2\phi_{n+1}(x) + x\phi_{n+2}(x) + \phi_{n+3}(x) . \tag{3.5}$$

Then

$$\overline{S}_0(x) = 1 \qquad \overline{S}_2(x) = x^2$$

$$\overline{S}_1(x) = x \qquad \overline{S}_3(x) = x^3 + x + 2 .$$

Referring to the table at the end of the paper we see that by (3.1)

$$\overline{S}_n(x) \equiv S_n(x) \pmod 2$$

for $n = 0, 1, 2,$ and 3. Therefore we see from (3.2), (3.4), and (3.5) that

$$\overline{S}_n(x) \equiv S_n(x) \pmod 2 \tag{3.6}$$

for all non-negative integers n.

From (3.3) we have with a little calculation that

$$\begin{aligned}\sum_{n=0}^{\infty} \phi_n(x)t^n &= \sum_{j=1}^{4} \frac{1}{1 - \alpha_j t} \\ &= \frac{t^3}{1 + t^3 + x^4t^4} \\ &= \sum_{n=0}^{\infty} t^n \sum_{3k+j+3=n} \binom{k}{j} x^{4j} ;\end{aligned}$$

therefore

$$\phi_n(x) = \sum_k \binom{k}{n - 3k - 3} x^{4(n-3k-3)} . \tag{3.7}$$

Combining (3.1), (3.5), (3.6) and (3.7) we have

$$\sum_{r=0}^{n} S_2(n,r)x^r \equiv \sum_k \binom{k-1}{n-3k-1} x^{4(n-3k)} + x\left\{\sum_k \binom{k}{n-3k-3} x^{4(n-3k-3)} + \sum_k \binom{k}{n-3k-1} x^{4(n-3k-1)}\right\} + x^2 \sum_k \binom{k}{n-3k-2} x^{4(n-3k-2)} + x^3 \sum_k \binom{k}{n-3k-3} x^{4(n-3k-3)} \pmod 2$$

Comparing coefficients we see that

$$(3.8)\quad \begin{cases} S_2(n,4j) \equiv \binom{r}{j-1} & (j = n-3r-3) \\ S_2(n,4j+1) \equiv \binom{r}{j} & (j = n-3r-3 \text{ or } n-3r-1) \\ S_2(n,4j+2) \equiv \binom{r}{j} & (j = n-3r-2) \\ S_2(n,4j+3) \equiv \binom{r}{j} & (j = n-3r-3), \end{cases}$$

where the modulus 2 is understood in each congruence.

Let $\theta_j(n)$ denote the number of odd $S_2(n,k)$, $0 \le k \le n$, with

$$k \equiv j \pmod 4 \quad (j = 0,1,2,3).$$

By the first congruence in (3.8) we see that

$$S_2(n+1,\ 4j+4) \equiv \binom{r}{j} \pmod 2 \quad (j = n-3r-3),$$

and hence

$$(3.9)\qquad \theta_0(n+1) = \theta_3(n).$$

Similarly since

$$S_2(n+2,\ 4j+4) \equiv \binom{r}{j} \pmod 2 \quad (j = n-3r-2),$$

it follows that

$$(3.10) \qquad \theta_0(n+2) = \theta_2(n) .$$

In a like manner we obtain

$$\begin{aligned}\theta_1(n) &= \theta_3(n) + \theta_2(n+1)\\ &= \theta_0(n+1) + \theta_0(n+3) ;\end{aligned}$$

the second equation follows from (3.9) and (3.10). Since all $\theta_j(n)$ may be expressed in terms of $\theta_0(n)$ it will suffice to determine the generating function for $\theta_0(n)$ alone.

Now by (3.8)

$$S_2(2n, 4j) \equiv \binom{r}{j-1} \pmod 2 \qquad (j = 2n - 3r - 3) .$$

From this it follows that

$$S_2(2n, 4j) \equiv 0 \pmod 2$$

unless

$$j \equiv r + 1 \pmod 2 .$$

Hence if we let

$$r = 2r' + s, \quad j - 1 = 2j' + s \quad (s = 0, 1) ,$$

then

$$S_2(2n, 4j) \equiv \binom{r'}{j'} \pmod 2 \quad (j' = n - 3r' - 2s - 2) ,$$

and therefore

(3.11)
$$\theta_0(2n) = \theta_2(n) + \theta_3(n-1)$$
$$= \theta_0(n+2) + \theta_0(n) \; .$$

Similarly, since

$$S_2(2n+1), 4j) \equiv \binom{r}{j-1} \pmod 2 \quad (j = 2n - 3r - 2) \; ,$$

we have

$$S_2(2n+1, 4j) \equiv 0 \pmod 2$$

unless

$$r \equiv j \equiv 1 \pmod 2 \; .$$

Letting

$$r = 2r' + 1, \quad j = 2j' + 1$$

we get

$$S_2(2n+1, 4j) \equiv \binom{r'}{j'} \pmod 2 \quad (j' = n - 3r' - 3) \; .$$

Therefore

(3.12)
$$\theta_0(2n+1) = \theta_3(n) = \theta_0(n+1) \; .$$

If we let

$$\omega(n) = \theta_0(n+4)$$

we obtain from (3.11) and (3.12) that

$$\omega(2n) = \omega(n) + \omega(n-2)$$

and

$$\omega(2n + 1) = \omega(n - 1) .$$

Since $\theta_0(1) = \theta_0(2) = \theta_0(3) = 0$, we have $\omega(n) = 0$ for $n < 0$, and these equations for $\omega(n)$ are valid for all $n = 0, 1, 2, \cdots$. Hence we have

$$\sum_{n=0}^{\infty} \omega(n)x^n = \sum_{n=0}^{\infty} \omega(2n)x^{2n} + \sum_{n=0}^{\infty} \omega(2n + 1)x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \omega(n)x^{2n} + \sum_{n=0}^{\infty} \omega(n - 2)x^{2n} + \sum_{n=0}^{\infty} \omega(n - 1)x^{2n+1}$$

$$= (1 + x^3 + x^4) \sum_{n=0}^{\infty} \omega(n)x^{2n}$$

$$= \prod_{n=0}^{\infty} (1 + x^{3\cdot 2^n} + x^{2^{n+2}}) ,$$

and the theorem is proved.

From this generating function we see that $\omega(n)$ also denotes the number of partitions

$$n = n_0 + n_1 \cdot 2 + n_2 \cdot 2^2 + n_3 \cdot 2^3 + \cdots \quad (n_j = 0, 3, 4) .$$

4. THE CASE $p = 3$

We shall now consider the above problem for the prime $p = 3$. Since the work is similar to that of Section 3, many of the details will be omitted.

From (2.4) we have

$$(4.1) \qquad S_2(n + 9, j) \equiv 2S_2(n + 3, j) + 2S_2(n + 1, j) + S_2(n, j - 9) \pmod 3 .$$

Therefore letting

(4.2) $$S_n(x) = \sum_{j=0}^{n} S_2(n,j)x^j ,$$

we have

(4.3) $$S_{n+9}(x) \equiv 2S_{n+3}(x) + 2S_{n+1}(x) + x^9 S_n(x) \pmod 3 .$$

Let $\alpha_1, \alpha_2, \cdots, \alpha_9$ be the roots of the equation

$$y^9 + y^3 + y - x^9 = 0$$

in $F[y]$, where $F = GF(3,x)$. Then if

$$\phi_n(x) = \sum_{j=1}^{9} \alpha_j^n ,$$

we see that

(4.4) $$\phi_0(x) = \phi_1(x) = \cdots = \phi_7(x) = 0, \quad \phi_8(x) = 1 .$$

Moreover

(4.5) $$\phi_{n+9}(x) = x^9\phi_n(x) - \phi_{n+1}(x) - \phi_{n+3}(x) ,$$

and hence

(4.6) $$\phi_9(x) = \phi_{10}(x) = \cdots = \phi_{13}(x) = \phi_{15}(x) = 0, \quad \phi_{14}(x) = \phi_{16}(x) = -1.$$

If we let

(4.7) $$\begin{cases} f_0(x) = S_0(x) + S_2(x) + S_8(x) \\ f_1(x) = S_1(x) + S_7(x) \\ f_2(x) = S_0(x) + S_6(x) \\ f_j(x) = S_{8-j}(x) & (j = 3, 4, \cdots, 8) \end{cases}$$

and

$$\overline{S}_n(x) = \sum_{j=0}^{8} f_j(x)\,\phi_{n+j}(x) \quad , \tag{4.8}$$

it is clear from (4.3), (4.4), $\cdots$, (4.8) that

$$\overline{S}_n(x) \equiv S_n(x) \pmod 3 \quad (n = 0, 1, 2, \cdots) . \tag{4.9}$$

As in Section 3 we see that

$$\sum_{n=0}^{\infty} \phi_n(x)t^n = \sum_{n=0}^{\infty} t^n \sum_{6k+8+r=n} (-1)^k \sum_{2j+h=r} \binom{k}{j}\binom{j}{h} (-1)^h x^{9h}$$

and hence

$$\phi_n(x) = \sum_k (-1)^{n+k} \sum_j \binom{k}{j} \binom{j}{n-6k-8-2j} x^{9(n-8-6k-2j)} . \tag{4.10}$$

By expanding (4.8), comparing coefficients and combining terms we have, for instance, from (4.2), (4.9), and (4.10) that

$$S_2(n+9,\ 9h+9) \equiv \sum_{j,k} (-1)^{n+k} \binom{k}{j}\binom{j}{h} \pmod 3$$

and

$$S_2(n+8,\ 9h+8) \equiv \sum_{j,k} (-1)^{n+k} \binom{k}{j}\binom{j}{h} \pmod 3 \quad ,$$

but

$$S_2(n+8,\ 9h+6) \equiv \sum_{j,k} (-1)^{n+k} \left\{ \binom{k}{j}\binom{j}{h} + \binom{k}{j+1}\binom{j+1}{h} \right\} \pmod 3 \quad ,$$

where the summations are over all nonnegative integers j and k such that $h = n - 6k - 2j$. The numbers $S_2(n, 9h + j)$ for $j = 0, 1, \cdots, 5$ are more complicated.

At this point the method employed in Section 3 seems to fail. As was mentioned in Section 1, the apparent difficulty in this case is the fact that the recurrence (4.1) is a four-term recurrence. If we consider the generalized Stirling number $S_3(n, r)$ and the prime $p = 2$ we again get a four-term recurrence; the development of the problem in this case is very similar to our work in the present section.

TABLE

Generalized Stirling Numbers of the Second Kind $S_2(n, r)$

n \ r	1	2	3	4	5	6	7	8
1	1							
2	2	1						
3	5	6	1					
4	15	32	12	1				
5	52	175	110	20	1			
6	203	1012	945	280	30	1		
7	877	6230	8092	3465	595	42	1	
8	4140	40819	70756	40992	10010	1120	56	1

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