ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problem.

Notice: If any of the readers have not received acknowledgment for their solutions in previous issues, the editor will acknowledge them after receipt of notification of such omissions.

H-123 Proposed by D. Lind, University of Virginia, Charlottesville, Virginia.

Prove

$$F_n = \sum_{m=0}^{n}\sum_{k=0}^{m} \mathcal{S}_n^{(m)} S_m^{(k)} F_k ,$$

where $S_r^{(s)}$ and $\mathcal{S}_r^{(s)}$ are Stirling numbers of the first and second kinds, respectively, and F_n is the n^{th} Fibonacci number.

H-124 Proposed by J. A. H. Hunter, Toronto, Canada

Prove the following identity:

$$F_{m+n}^2 L_{m+n}^2 - F_m^2 L_m^2 = F_{2n} F_{2(m+n)}$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

H-125 Proposed by Stanley Rabinowitz, Far Rockaway, New York

Define a sequence of positive integers to be left-normal if given any string of digits, there exists a member of the given sequence beginning with this string of digits, and define the sequence to be right-normal if there exists a member of the sequence ending with this string of digits.

Show that the sequences whose n^{th} terms are given by the following are left-normal but not right-normal.

a) $P(n)$, where $P(x)$ is a polynomial function with integral coefficients.

b) P_n, where P_n is the n^{th} prime.

c) $n!$

d) F_n, where F_n is the n^{th} Fibonacci number.

SOLUTIONS

EUREKA!

H-59 Proposed by D. W. Robinson, Brigham Young University, Provo, Utah.

Show that if $m > 2$, then the period of the Fibonacci sequence $0, 1, 2, 3, \cdots, F_n, \cdots$ reduced modulo m is twice the least positive integer, n, such that

$$F_{n+1} \equiv (-1)^n F_{n-1} \pmod{m}$$

Solution by James E. Desmond, Tallahassee, Florida.

Let s be the period of the Fibonacci sequence modulo m. Then by defin-intion, s is the least positive integer such that

$$(1) \qquad F_{s-1} \equiv 1 \pmod{m} \text{ and } F_s \equiv 0 \pmod{m}.$$

By the well-known formula

$$F_{s+1}F_{s-1} - F_s^2 = (-1)^s$$

We find that $1 \equiv (-1)^s \pmod m$ which implies, since $m > 2$, that $s = 2t$ for some positive integer t. It is easily verified that

$$F_{2t-1} = F_t L_{t-1} + (-1)^t = F_{t-1}L_t + (-1)^{t+1} \quad (2)$$

Since $s = 2t$ we have by (1) and (2) that

$$F_t L_{t-1} \equiv 0 \pmod m \text{ if } t \text{ is even, and} \quad (3)$$

$$F_{t-1}L_t \equiv 0 \pmod m \text{ if } t \text{ is odd.} \quad (4)$$

It is well known that

$$F_{2t} = F_t L_t \text{ , and} \quad (5)$$

$$(L_{t-1}, L_t) = (F_{t-1}, F_t) = 1 \; . \quad (6)$$

Thus by (1), (3), (4), (5), and (6) we have

$$F_t \equiv 0 \pmod m \text{ if } t \text{ is even, and}$$
$$L_t \equiv 0 \pmod m \text{ if } t \text{ is odd, i.e.,}$$
$$F_{t+1} + (-1)^{t+1}F_{t-1} \equiv 0 \pmod m \; .$$

Now, let n be the least positive integer such that $F_{n+1} + (-1)^{n+1}F_{n-1} \equiv 0 \pmod m$ and we obtain $n \le t$. We also find that $F_n \equiv 0 \pmod m$ if n is even, and $L_n \equiv 0 \pmod m$ if n is odd. Thus by (2) we have, $F_{2n-1} \equiv 1 \pmod m$ and by (5), $F_{2n} \equiv 0 \pmod m$. Since s is the period modulo m, **it** follows by definition that $2t = s \le 2n$. Hence $n = t$.

RESTRICTED UNFRIENDLY SUBSETS

H-75 Proposed by Douglas Lind, University of Virginia, Charlottesville, Virginia.

Show that the number of distinct integers with one element n, all other elements less than n and not less than k, and such that no two consecutive integers appear in the set is F_{n-k+1}.

Solution by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pa.

Since each admissible set of integers must contain n, any given admissible set is uniquely determined by specifying which of the remaining $n - k - 1$ integers $(k, k+1, k+2, \cdots, n-2)$ are included in the set. (Note that the integer $n - 1$ cannot be included since n is in each set and consecutive integers are not permitted.) For each set, this information can be given concisely by a sequence of $n - k - 1$ binary digits, using a 1 in the m^{th} place $(m = 1, 2, \cdots, n-k-1)$ if the integer $k + m - 1$ is included in the set and 0 in the m^{th} place otherwise.

If we require additionally that the terms of each such binary sequence $(\alpha_1, \alpha_2, \cdots, \alpha_{n-k-1})$ satisfy $\alpha_i \alpha_{i+1} = 0$ for $i = 1, 2, \cdots, n-k-2$, then this requirement is equivalent to the condition that no two consecutive integers appear in the corresponding set. But the number of distinct binary sequences of length $n - k - 1$ satisfying $\alpha_i \alpha_{i+1} = 0$ for $i \geq 1$ is known to be $F_{(n-k-1)+2} = F_{n-k+1}$ as required. [See The Fibonacci Quarterly, Vol. 2, No. 3, pp. 166-167 for a proof using Zeckendorf's Theorem.]

FIBONOMIAL COEFFICIENT GENERATORS

H-78 Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

(i) Show

$$\frac{x^{n-1}}{(1-x)^n} = \sum_{m=0}^{\infty} \binom{m}{n-1} x^m, \qquad (n \geq 1)$$

where $\binom{m}{n}$ are the binomial coefficients.

(ii) Show

$$\frac{x}{(1 - x - x^2)} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ 1 \end{bmatrix} x^m ,$$

$$\frac{x^2}{(1 - 2x - 2x^2 + x^3)} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ 2 \end{bmatrix} x^m,$$

$$\frac{x^3}{(1 - 3x - 6x^2 + 3x^3 + x^4)} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ 3 \end{bmatrix} x^m ,$$

where $\begin{bmatrix} m \\ n \end{bmatrix}$ are the Fibonomial coefficients as in H-63, April 1965, Fibonacci Quarterly and H-72 of Dec. 1965, Fibonacci Quarterly.

The generalization is: Let

$$f(x) = \sum_{h=0}^{k} (-1)^{h(h+1)/2} \begin{bmatrix} k \\ h \end{bmatrix} x^h ,$$

then show

$$\frac{x^{k-1}}{f(x)} = \sum_{m=0}^{\infty} \begin{bmatrix} m \\ k-1 \end{bmatrix} x^m , \quad (k \geq 1) .$$

Solution by L. Carlitz, Duke University.

(i) This is a special case of the binomial theorem.

(ii) The general resultscan be viewed as the q-analog of (i), namely

$$\prod_{j=0}^{k-1} (1 - q^j x)^{-1} = \sum_{j-0}^{\infty} \left\{ \begin{matrix} k+j-1 \\ j \end{matrix} \right\} x^j ,$$

where

$$\left\{ \begin{matrix} k+j-1 \\ j \end{matrix} \right\} = \frac{(1 - q^k)(1 - q^{k+1}) \cdots (1 - q^{k+j-1})}{(1 - q)(1 - q^2) \cdots (1 - q^j)} .$$

We shall also need

$$\prod_{j=0}^{k-1} (1 - q^j x) = \sum_{j=0}^{k} (-1)^j q^{\frac{1}{2}j(j-1)} \left\{ {k \atop j} \right\} x^j$$

Now take $q = \beta/\alpha$, $\alpha = \frac{1}{2}(1 + \sqrt{5})$, $\beta = \frac{1}{2}(1 - \sqrt{5})$. Then

$$\left\{ {k \atop j} \right\} \longrightarrow \alpha^{-j(k-j)} \frac{F_k F_{k-1} \cdots F_{k-j+1}}{F_1 F_2 \cdots F_j} = \alpha^{-(k-1)j} \begin{bmatrix} k \\ j \end{bmatrix}.$$

(Compare "Generating Functions for Powers of Certain Sequences of Numbers," Duke Mathematical Journal, Vol. 29 (1962), pp. 521-538, particularly p. 530.)

Since

$$\begin{aligned} (-1)^j \left(\frac{\beta}{\alpha} \right)^{\frac{1}{2}j(j-1)} \alpha^{-j(k-j)} &= (-1)^j (-\alpha^{-2})^{\frac{1}{2}j(j-1)} \alpha^{-j(k-j)} \\ &= (-1)^{\frac{1}{2}j(j+1)} \alpha^{-j(k-1)} \quad , \end{aligned}$$

we get, after replacing x by $\alpha^{k-1}x$, the identity

$$\left\{ \sum_{j=0}^{k-1} (-1)^{\frac{1}{2}j(j+1)} \begin{bmatrix} k \\ j \end{bmatrix} x^j \right\}^{-1} = \sum_{j=0}^{\infty} \begin{bmatrix} k + j - 1 \\ j \end{bmatrix} x^j = \sum_{j=0}^{\infty} \begin{bmatrix} k + j - 1 \\ k - 1 \end{bmatrix} x^j$$

A FOURTH-POWER FORMULA

H-79 Proposed by J. A. H. Hunter, Toronto, Ontario, Canada.

Show

$$F_{n+1}^4 + F_n^4 + F_{n-1}^4 = 2\left[2F_n^2 + (-1)^n\right]^2 .$$

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

From the well-known identity,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

we have,

$$\begin{aligned}
2\left[2F_n^2 + (-1)^n\right]^2 &= 2\left[F_{n-1}F_{n+1} + F_n^2\right]^2 \\
&= F_n^4 + F_n^4 + 2F_{n-1}^2F_{n+1}^2 + 4F_n^2F_{n-1}F_{n+1} \\
&= F_n^4 + F_n^2(F_n^2 + 4F_{n-1}F_{n+1}) + 2F_{n-1}^2F_{n+1}^2 \\
&= F_n^4 + F_n^2\left[(F_{n+1} - F_{n-1})^2 + 4F_{n-1}F_{n+1}\right] + 2F_{n-1}^2F_{n+1}^2 \\
&= F_n^4 + (F_{n+1} - F_{n-1})^2(F_{n+1} + F_{n-1})^2 + 2F_{n-1}^2F_{n+1}^2 \\
&= F_n^4 + (F_{n+1}^2 - F_{n-1}^2)^2 + 2F_{n-1}^2F_{n+1}^2 \\
&= F_n^4 + F_{n+1}^4 + F_{n-1}^4
\end{aligned}$$

Hence,

$$F_{n+1}^4 + F_n^4 + F_{n-1}^4 = 2\left[2F_n^2 + (-1)^n\right]^2$$

Also solved by Thomas Dence, F. D. Parker, and L. Carlitz.

A PLEASANT SURPRISE

H-80 Proposed by J. A. H. Hunter, Toronto, Canada, and Max Rumney, London, England (corrected).

Show

$$\sum_{r=0}^{n} \binom{n}{r} F_{r+2}^2 = \sum_{r=0}^{n} \binom{n-1}{r} F_{2r+5} .$$

Solution by L. Carlitz, Duke University

This is correct for $n = 0$, so we assume that $n > 0$. Since

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \alpha = \tfrac{1}{2}(1 + \sqrt{5}), \ \beta = \tfrac{1}{2}(1 - \sqrt{5}),$$

we have

$$5\sum_{r=0}^{n} \binom{n}{r} F_{r+2}^2 = \sum_{r=0}^{n} \binom{n}{r}\left[\alpha^{2r+4} - 2(-1)^r + \beta^{2r+4}\right]$$

$$= \alpha^4(\alpha^2 + 1)^n + \beta^4(\beta^2 + 1)^n .$$

On the other hand

$$\sum_{r=0}^{n-1} \binom{n-1}{r} F_{2r+5} = \frac{1}{\alpha - \beta} \sum_{r=0}^{n-1} \binom{n-1}{r} (\alpha^{2r+5} - \beta^{2r+5})$$

$$= \frac{\alpha^5(\alpha^2 + 1)^{n-1} - \beta^5(\beta^2 + 1)^{n-1}}{\alpha - \beta}$$

Thus it suffices to show that

$$\alpha^4(\alpha^2 + 1)^{n-1} + \beta^4(\beta^2 + 1)^{n-1} = (\alpha - \beta)\left[\alpha^5(\alpha^2 + 1)^{n-1} - \beta^5(\beta^2 + 1)^{n-1}\right].$$

The right side is equal to

$$\alpha^6(\alpha^2 + 1)^{n-1} + \beta^6(\beta^2 + 1)^{n-1} + \alpha^4(\alpha^2 + 1)^{n-1} + \beta^4(\beta^2 + 1)^{n-1}$$

$$= \alpha^4(\alpha^2 + 1)^n + \beta^4(\beta^2 + 1)^n .$$

<u>Remark.</u> More generally we have

$$\sum_{r=0}^{n} \binom{n}{r} F^2_{kr+2k} = F_k \sum_{r=0}^{n-1} \binom{n-1}{r} F_{2kr+5k}$$

for k odd and $n > 0$.

Also solved by M. N. S. Swamy, F. D. Parker, and Douglas Lind.

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A NOTE OF JOY

We have received with great pleasure the announcement of the forthcoming Journal of Recreational Mathematics under the editorship of Joseph S. Madachy. Volume 1, Number 1 is to appear in January, 1968. The journal "will deal with the lighter side of mathematics, that side devoted to the enjoyment of mathematics; it will depart radically from textbook problems and discussions and will present original, thought-provoking, lucid and exciting articles which will appeal to both students and teachers in the field of mathematics." The journal will feature authoritative articles concerning number theory, geometric constructions, dissections, paper folding, magic squares, and other number phenomena. There will be problems and puzzles, mathematical biographies and histories. Subscriptions for the Journal of Recreational Mathematics are handled by Greenwood Periodicals, Inc., 211 East 43rd St., New York, N. Y. 10017. We wish this valuable and important journal all possible success. H.W.E.

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