ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-124 Proposed by J. H. Butchart, Northern Arizona University, Flagstaff, Ariz.

Show that

\[ \sum_{i=0}^{\infty} \left( \frac{a_i}{2^i} \right) = 4, \]

where

\[ a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \ldots \]

are the Fibonacci numbers.

B-125 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Is

\[ \sum_{k=3}^{n} \frac{1}{F_k} \]

ever an integer for \( n \geq 3 \)? Explain,
Dec. 1967 ELEMENTARY PROBLEMS AND SOLUTIONS

B-126 Proposed by J. A. H. Hunter, Toronto, Canada

Each distinct letter in this alphametic stands, of course, for a particular and different digit. The advice is sound, for our FQ is truly prime. What do you make of it all?

\[
\begin{array}{c}
R \ E \ A \ D \\
F \ Q \\
R \ E \ A \ D \\
F \ Q \\
D \ E \ A \ R
\end{array}
\]

B-127 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Show that

\[
2^n L_n \equiv 2 \pmod{5},
\]
\[
2^n F_n \equiv 2n \pmod{5}.
\]


Let \( f_n \) be the generalized Fibonacci sequence with \( f_1 = a, f_2 = b \), and \( f_{n+1} = f_n + f_{n-1} \). Let \( g_n \) be the associated generalized Lucas sequence defined by \( g_n = f_{n-1} + f_{n+1} \). Also let \( S_n = f_1 + f_2 + \ldots + f_n \). It is true that \( S_4 = g_4 \) and \( S_8 = 3g_8 \). Generalize these formulas.

B-129 Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.

For a given positive integer, \( k \), find

\[
\lim_{n \to \infty} \frac{(F_{n+k}/L_n)}{n}.
\]

B-130 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let coefficients \( c_j(n) \) be defined by
and show that

$$\sum_{j=0}^{2n} [c_j(n)]^2 = c_{2n}(2n) .$$

Generalize to

$$(1 + x + x^2 + \cdots + x^k)^n .$$

B-131 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Prove that for $m$ odd

$$\frac{L_{n-m} + L_{n+m}}{F_{n-m} + F_{n+m}} = \frac{5F_n}{L_n}$$

and for $m$ even

$$\frac{F_{n-m} + F_{n+m}}{F_{n-m} + L_{n+m}} = \frac{F_n}{L_n} .$$

**SOLUTIONS**

**Note:** In the last issue, we inadvertently omitted M. N. S. Swamy from the solvers of B-100, B-101, and B-104.

**FIBONACCI–LUCAS ADDITION FORMULAS**

B-106 Proposed by H. H. Ferns, Victoria, B.C., Canada.

Prove the following identities:
AND SOLUTIONS

\[
2F_{1+j} = F_j L_1 + F_1 L_j, \\
2L_{1+j} = L_j L_1 + 5F_1 F_j.
\]

Solution by Charles R. Wall, University of Tennessee, Knoxville, Tennessee.

From the Binet formulas we have

\[
F_j L_1 + F_1 L_j = \frac{1}{\sqrt{5}} \left\{ (a^i \cdot b^i) (a^j \cdot b^j) + (a^i \cdot b^j) (a^j \cdot b^i) \right\}
\]

\[
= \frac{2}{\sqrt{5}} (a^{i+j} - b^{i+j}) = 2F_{1+j},
\]

and

\[
L_j L_1 + 5F_1 F_j = (a^i + b^i) (a^j + b^j) + (a^i - b^i) (a^j - b^j)
\]

\[
= 2(a^{i+j} + b^{i+j}) = 2L_{1+j}.
\]


AN APPROXIMATION

B-107 Proposed by Robert S. Seamons, Yakima Valley College, Yakima, Wash.

Let \( M_n \) and \( G_n \) be respectively the \( n \)th terms of the sequences (of Lucas and Fibonacci) for which \( M_n = M_{n-1}^2 - 2 \), \( M_1 = 3 \), and \( G_n = G_{n-1} + G_{n-2} \), \( G_1 = 1 \), \( G_2 = 2 \). Prove that

\[
M_n = 1 + \left\lfloor \sqrt{5} G_m \right\rfloor,
\]

where \( m = 2^n - 1 \) and \( \left\lfloor x \right\rfloor \) is the greatest integer function.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.
In standard notation we have \( M_n = L_n \) and \( G_n = F_{n+1} \), where \( F_n \) and \( L_n \) are the \( n \)th Fibonacci and Lucas numbers, respectively. The problem then becomes to show

\[
L_{2n} = \left[ 1 + \sqrt{5} F_n \right],
\]

which follows immediately from Problem B-89.


**GENERALIZED FIBONACCI NUMBERS**

B-108 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let \( u_1 = p, \ u_2 = q, \) and \( u_{n+2} = u_{n+1} + u_n. \) Also let \( S_n = u_1 + u_2 + \cdots + u_n. \) It is true that \( S_6 = 4u_4 \) and \( S_{10} = 11u_7. \) Generalize these formulas.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

The problem should read \( S_6 = 4u_4. \) The fact that

\[
\sum_{i=1}^{4k-2} u_i = L_{2k-2} u_{2k+1}
\]


**SECOND-ORDER DIFFERENCE EQUATION**

B-109 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let \( r \) and \( s \) be the roots of the quadratic equation \( x^2 - px - q = 0, \) \((r \neq s)\). Let \( U_n = (r^n - s^n)/(r - s) \) and \( V_n = r^n + s^n. \) Show that
\[
V_n = U_{n+1} + qU_{n-1}.
\]

Solution by Charles W. Trigg, San Diego, California.

\[q = -rs,
\]

\[
U_{n+1} + qU_{n-1} = \frac{(r^{n+1} - s^{n+1})}{(r - s)} + \frac{(-rs)(r^{n-1} - s^{n-1})}{(r - s)}\\ = \left(\frac{r^n(r - s) + s^n(r - s)}{r - s}\right)\\ = V_n.
\]


**AN INFINITE SERIES EQUALITY**

B-110 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

\[
\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2n+1}}
\]

Solution by the proposer.

\[
F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad \alpha = \frac{1}{2} (1 + \sqrt{5}), \quad \beta = \frac{1}{2} (1 - \sqrt{5}).
\]

Then
\[
\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = (\alpha - \beta) \sum_{n=0}^{\infty} \frac{1}{\alpha^{2n+1} - \beta^{2n+1}}
\]

\[
= (\alpha - \beta) \sum_{n=0}^{\infty} \frac{1}{\alpha^{2n+1} - 1} \cdot \frac{1}{1 + \alpha^{-2(2n+1)}}
\]

\[
= (\alpha - \beta) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} (-1)^r \alpha^{-(2r+1)(2n+1)}
\]

\[
= (\alpha - \beta) \sum_{r=0}^{\infty} \frac{(-1)^r}{\alpha^{-2r+1} - 1}
\]

\[
= (\alpha - \beta) \sum_{r=0}^{\infty} \frac{(-1)^r}{\alpha^{-2r+1} + \beta^{2r+1}}
\]

\[
= \sqrt{5} \sum_{r=0}^{\infty} \frac{(-1)^r}{1 - 2r+1}
\]

ANOTHER SERIES EQUALITY


Show that

\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{L_{4n+2}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{1}{L_{4n+2}}
\]

Solution by the proposer.
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{F_{2(2n+i)}} = (\alpha - \beta) \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha^{2(2n+i)} - \beta^{2(2n+i)}}
\]

\[
= (\alpha - \beta) \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha^{2(2n+i)}} \frac{1}{1 - \alpha^{-4(2n+i)}}
\]

\[
= (\alpha - \beta) \sum_{n=0}^{\infty} (-1)^n \sum_{r=0}^{\infty} \alpha^{-2(2r+1)(2n+i)}
\]

\[
= (\alpha - \beta) \sum_{r=0}^{\infty} \alpha^{-2(2r+1)} \frac{1}{1 + \alpha^{-4(2r+1)}}
\]

\[
= (\alpha - \beta) \sum_{r=0}^{\infty} \frac{1}{\alpha^{2(2r+1)} + \beta^{2(2r+1)}}
\]

\[
= \sqrt{5} \sum_{r=0}^{\infty} \frac{1}{I_{2(2r+1)}}
\]

****

NOTICE TO ALL SUBSCRIBERS!!

Please notify the Managing Editor AT ONCE of any address change. The Post Office Department, rather than forwarding magazines mailed third class, sends them directly to the dead-letter office. Unless the addressee specifically requests the Fibonacci Quarterly to be forwarded at first class rates to the new address, he will not receive it. (This will usually cost about 30 cents for first-class postage.) If possible, please notify us AT LEAST THREE WEEKS PRIOR to publication dates: February 15, April 15, October 15, and December 15.