ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-124 Proposed by J. H. Butchart, Northern Arizona University, Flagstaff, Ariz.

Show that

$$\sum_{i=0}^{\infty}(a_i / 2^i) = 4 ,$$

where

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \cdots$$

are the Fibonacci numbers.

B-125 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Is

$$\sum_{k=3}^{n} \frac{1}{F_k}$$

ever an integer for $n \geq 3$? Explain.

B-126 Proposed by J. A. H. Hunter, Toronto, Canada

Each distinct letter in this alphametic stands, of course, for a particular and different digit. The advice is sound, for our FQ is truly <u>prime</u>. What do you make of it all?

$$\begin{array}{r} \mathrm{R\ E\ A\ D} \\ \mathrm{F\ Q} \\ \mathrm{R\ E\ A\ D} \\ \underline{\mathrm{F\ Q}} \\ \mathrm{D\ E\ A\ R} \end{array}$$

B-127 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Show that

$$2^n L_n \equiv 2 \pmod 5 ,$$
$$2^n F_n \equiv 2n \pmod 5 .$$

B-128 Proposed by M. N. S. Swamy, Nova Scotia Tech. College, Halifax, Canada.

Let f_n be the generalized Fibonacci sequence with $f_1 = a$, $f_2 = b$, and $f_{n+1} = f_n + f_{n-1}$. Let g_n be the associated generalized Lucas sequence defined by $g_n = f_{n-1} + f_{n+1}$. Also let $S_n = f_1 + f_2 + \cdots + f_n$. It is true that $S_4 = g_4$ and $S_8 = 3g_6$. Generalize these formulas.

B-129 Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.

For a given positive integer, k, find

$$\lim_{n\to\infty}(F_{n+k}/L_n) .$$

B-130 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let coefficients $c_j(n)$ be defined by

$$(1 + x + x^2)^n = c_0(n) + c_1(n)x + c_2(n)x^2 + \cdots + c_{2n}(n)x^{2n}$$

and show that

$$\sum_{j=0}^{2n} \left[c_j(n)\right]^2 = c_{2n}(2n) .$$

Generalize to

$$(1 + x + x^2 + \cdots + x^k)^n .$$

B-131 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Prove that for m odd

$$\frac{L_{n-m} + L_{n+m}}{F_{n-m} + F_{n+m}} = \frac{5F_n}{L_n}$$

and for m even

$$\frac{F_{n-m} + F_{n+m}}{L_{n-m} + L_{n+m}} = \frac{F_n}{L_n} .$$

SOLUTIONS

Note: In the last issue, we inadvertently omitted M. N. S. Swamy from the solvers of B-100, B-101, and B-104.

FIBONACCI-LUCAS ADDITION FORMULAS

B-106 Proposed by H. H. Ferns, Victoria, B.C., Canada.

Prove the following identities:

$$2F_{i+j} = F_iL_j + F_jL_i \quad ,$$
$$2L_{i+j} = L_iL_j + 5F_iF_j \quad .$$

Solution by Charles R. Wall, University of Tennessee, Knoxville, Tennessee.

From the Binet formulas we have

$$F_iL_j + F_jL_i = \frac{1}{\sqrt{5}}\left\{(\alpha^i - \beta^i)(\alpha^j + \beta^j) + (\alpha^j - \beta^j)(\alpha^i + \beta^i)\right\}$$
$$= \frac{2}{\sqrt{5}}(\alpha^{i+j} - \beta^{i+j}) = 2F_{i+j} \quad ,$$

and

$$L_iL_j + 5F_iF_j = (\alpha^i + \beta^i)(\alpha^j + \beta^j) + (\alpha^i - \beta^i)(\alpha^j - \beta^j)$$
$$= 2(\alpha^{i+j} + \beta^{i+j}) = 2L_{i+j} \quad .$$

Also solved by John H. Biggs, Douglas Lind, William C. Lombard, C. B. A. Peck, A. G. Shannon, M. N. S. Swamy, John Wessner, David Zeitlin, and the proposer.

AN APPROXIMATION

B-107 Proposed by Robert S. Seamons, Yakima Valley College, Yakima, Wash.

Let M_n and G_n be respectively the n^{th} terms of the sequences (of Lucas and Fibonacci) for which $M_n = M_{n-1}^2 - 2$, $M_1 = 3$, and $G_n = G_{n-1} + G_{n-2}$, $G_1 = 1$, $G_2 = 2$. Prove that

$$M_n = 1 + \left[\sqrt{5}\,G_m\right] \quad ,$$

where $m = 2^n - 1$ and $[x]$ is the greatest integer function.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

In standard notation we have $M_n = L_{2^n}$ and $G_n = F_{n+1}$, where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively. The problem then becomes to show

$$L_{2^n} = \left[1 + \sqrt{5}\, F_{2^n}\right] \quad ,$$

which follows immediately from Problem B-89.

Also solved by William C. Lombard, C. B. A. Peck, A. G. Shannon, David Zeitlin, and the proposer.

GENERALIZED FIBONACCI NUMBERS

B-108 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let $u_1 = p$, $y_2 = q$, and $u_{n+2} = u_{n+1} + u_n$. Also let $S_n = u_1 + u_2 + \cdots + u_n$. It is true that $S_6 = 4u_4$ and $S_{10} = 11u_7$. Generalize these formulas.

Solution by Douglas Lind, University of Virginia, Charlottesville, Va.

The problem should read $S_6 = 4u_5$. The fact that

$$\sum_{i=1}^{4k-2} u_i = L_{2k-1}u_{2k+1}$$

where L_n is the n^{th} Lucas number, appears in the solution of Problem 4272, American Math. Monthly, Vol. 56 (1949), p. 421.

Also solved by William C. Lombard, F. D. Parker, C. B. A. Peck, A. G. Shannon, M. N. S. Swamy, Charles R. Wall, David Zeitlin, and the proposer.

SECOND-ORDER DIFFERENCE EQUATION

B-109 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Let r and s be the roots of the quadratic equation $x^2 - px - q = 0$, $(r \neq s)$. Let $U_n = (r^n - s^n)/(r - s)$ and $V_n = r^n + s^n$. Show that

$$V_n = U_{n+1} + qU_{n-1} .$$

Solution by Charles W. Trigg, San Diego, California.

$$q = -rs,$$

so

$$\begin{aligned} U_{n+1} + qU_{n-1} &= (r^{n+1} - s^{n+1})/(r - s) + (-rs)(r^{n-1} - s^{n-1})/(r - s) \\ &= [r^n(r - s) + s^n(r - s)]/(r - s) \\ &= V_n . \end{aligned}$$

Also solved by Harold Don Allen, J. H. Biggs, Douglas Lind, William C. Lombard, F. D. Parker, C. B. A. Peck, M. N. S. Swamy, Charles R. Wall, John Wessner, David Zeitlin, and the proposer.

AN INFINITE SERIES EQUALITY

B-110 Proposed by L. Carlitz, Duke University, Durham, N. Carolina.

Show that

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2n+1}}$$

Solution by the proposer.

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \; L_n = \alpha^n + \beta^n, \; \alpha = \tfrac{1}{2}(1 + \sqrt{5}), \; \beta = \tfrac{1}{2}(1 - \sqrt{5}) .$$

Then

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}} = (\alpha - \beta) \sum_{n=0}^{\infty} \frac{1}{\alpha^{2n+1} - \beta^{2n+1}}$$

$$= (\alpha - \beta) \sum_{n=0}^{\infty} \frac{1}{\alpha^{2n+1}} \; \frac{1}{1 + \alpha^{-2(2n+1)}}$$

$$= (\alpha - \beta) \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} (-1)^r \alpha^{-(2r+1)(2n+1)}$$

$$= (\alpha - \beta) \sum_{r=0}^{\infty} \frac{(-1)^r \alpha^{-2r-1}}{1 - \alpha^{-2(2r+1)}}$$

$$= (\alpha - \beta) \sum_{r=0}^{\infty} \frac{(-1)^r}{\alpha^{2r+1} - \alpha^{-2r-1}}$$

$$= (\alpha - \beta) \sum_{r=0}^{\infty} \frac{(-1)^r}{\alpha^{2r+1} + \beta^{2r+1}}$$

$$= \sqrt{5} \sum_{r=0}^{\infty} \frac{(-1)^r}{L_{2r+1}} \quad .$$

ANOTHER SERIES EQUALITY

B-111 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Show that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{F_{4n+2}} = \sqrt{5} \sum_{n=0}^{\infty} \frac{1}{L_{4n+2}} \quad .$$

Solution by the proposer.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{F_{2(2n+1)}} = (\alpha - \beta) \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha^{2(2n+1)} - \beta^{2(2n+1)}}$$

$$= (\alpha - \beta) \sum_{n=0}^{\infty} \frac{(-1)^n}{\alpha^{2(2n+1)}} \frac{1}{1 - \alpha^{-4(2n+1)}}$$

$$= (\alpha - \beta) \sum_{n=0}^{\infty} (-1)^n \sum_{r=0}^{\infty} \alpha^{-2(2r+1)(2n+1)}$$

$$= (\alpha - \beta) \sum_{r=0}^{\infty} \frac{\alpha^{-2(2r+1)}}{1 + \alpha^{-4(2r+1)}}$$

$$= (\alpha - \beta) \sum_{r=0}^{\infty} \frac{1}{\alpha^{2(2r+1)} + \beta^{2(2r+1)}}$$

$$= \sqrt{5} \sum_{r=0}^{\infty} \frac{1}{L_{2(2r+1)}} .$$

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