THE BRACKET FUNCTION, q-BINOMIAL COEFFICIENTS, AND SOME NEW STIRLING NUMBER FORMULAS*

H. W. GOULD
W. Virginia University, Morgantown, W. Va.

TO PROFESSOR LEONARD CARLITZ ON HIS SIXTIETH BIRTHDAY, 26 December, 1967.

In a recent paper [3] the author proved that the binomial coefficient and the bracket function ($[x]$ = greatest integer $\le x$) are related by

$$(1) \qquad \binom{n}{k} = \sum_{j=k}^{n} \left[\frac{n}{j}\right] R_k(j)$$

and

$$(2) \qquad \left[\frac{n}{k}\right] = \sum_{j=k}^{n} \binom{n}{j} A_k(j) ,$$

where

(3) $R_k(j)$ = Number of compositions of j into k relatively prime positive summands,

$$= \sum_{\substack{a_1+\cdots+a_k=j \\ (a_1,\cdots,a_k)=1}} 1 ,$$

$$= \sum_{d|j} \binom{d-1}{k-1} \mu(j/d) ,$$

and

$$(4) \qquad A_k(j) = \sum_{d=k}^{j} (-1)^{j-d}\binom{j}{d}\left[\frac{d}{k}\right] = \sum_{1\le d\le j/k} (-1)^{j-kd} \binom{j-1}{kd-1} .$$

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Moreover, the fact that the numbers R and A are orthogonal proved the elegant general result that for any two sequences $f(n, k)$, $g(n, k)$, then

$$f(n, k) = \sum_{j=k}^{n} g(n, j) R_k(j) \tag{5}$$

if and only if

$$g(n, k) = \sum_{j=k}^{n} f(n, j) A_k(j) \; . \tag{6}$$

Notice that (5) and (6) do not imply (1) and (2); one at least of the special expansions must be proved before the inverse relation follows from (5)-(6).

Finally, it was found that R and A satisfy the congruences

$$\begin{aligned} R_k(j) &\equiv 0 \pmod{k} \\ A_k(j) &\equiv 0 \pmod{k} \end{aligned} \tag{7}$$

for all natural numbers $j \geq k + 1$ if and only if k is a prime.

These congruences, together with the fact that $R_k(k) = A_k(k) = 1$ then showed that either of (1) and (2) implies that

$$\binom{n}{k} \equiv \left[\frac{n}{k}\right] \pmod{k} \quad (k \geq 2) \tag{8}$$

for all natural numbers n if and only if k is a prime.

Naturally, similar congruences are implied for any f and g which satisfy the pair (5)-(6).

Now it is natural to look for an extension of these results to the more general situation where $\binom{n}{k}$ is replaced by the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=1}^{k} \frac{q^{n-j+1} - 1}{q^j - 1} , \quad \begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1 \; . \tag{9}$$

In the limiting case $q = 1$ these become ordinary binomial coefficients. This is the motivation for the present paper. Ordinarily we omit the subscript q unless we wish to emphasize the base used.

We follow the terminology in [2] and, since that paper is intimately connected with the results below, the reader is referred there for detailed statements and for further references to the literature. Cf. also [1].

In the present paper we exhibit q-analogs of expansions (1) and (2) in terms of q-extensions of R and A. Moreover, the generating functions for $R_k(j, q)$ and $A_k(j, q)$ prove their orthogonal nature so that we obtain an elegant and direct generalization of the inverse pair (5)-(6) to the q-coefficient case. By consideration of the expressions

$$\sum_{j=k}^{n} R_k(j, q) A_j(n, p) \ , \qquad \sum_{j=k}^{n} A_k(j, q) R_j(n, p) \ , \qquad q \neq p \ ,$$

we are then able to obtain new expressions for q-Stirling numbers of first and second kind, with the ordinary Stirling numbers as limiting cases.

Our emphasis is on the various series expansions involving R and A and a detailed study of arithmetic properties will be left for a separate paper.

The principal results developed here are embodied in Theorems 1-16. Special attention is called to 1, 2, and 6. A few arithmetic results also appear.

We begin by generalizing (2). Put

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q A_k(j, q), \qquad k \geq 1 .$$

Now, inverse relations (7.3)-(7.4) in [2] may be stated in the form

$$(10) \qquad F(n) = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} f(j)$$

if and only if

$$f(n) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} F(j) . \tag{11}$$

Thus

$$A_k(n, q) = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} \begin{bmatrix} j \\ k \end{bmatrix} .$$

In this sum the bracketed term is zero for $0 \leq j < k$ so that the index j need range only from k to n, and it is then also clear that $A_k(n, q) = 0$ for $n < k$. Moreover $A_k(k, q) = 1$ for all $k \geq 1$ and any q. Evidently we have proved

Theorem 1. The q-binomial coefficient expansion of the bracket function is

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix} A_k(j, q) = \begin{bmatrix} n \\ k \end{bmatrix} + \sum_{j=k+1}^{n} \begin{bmatrix} n \\ j \end{bmatrix} A_k(j, q) , \tag{12}$$

where

$$A_k(j, q) = \sum_{d=k}^{j} (-1)^{j-d} \begin{bmatrix} j \\ d \end{bmatrix} q^{(j-d)(j-d-1)/2} \begin{bmatrix} d \\ k \end{bmatrix} \tag{13}$$

$$= q^{j(j-1)/2} \sum_{d=k}^{j} (-1)^{j-d} \begin{bmatrix} j \\ d \end{bmatrix}_p p^{d(d-1)/2} \begin{bmatrix} d \\ k \end{bmatrix}$$

with $pq = 1$. Cf. also Theorem 15.

The indicated second form of (13) follows from the reciprocal transformation [2]

$$\begin{bmatrix} n \\ k \end{bmatrix}_p = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{for} \quad pq = 1 .$$

The sum may be written also as in the second form of (4) above. See Theorem 15.

The ease of finding (12) suggests that it should not be difficult to invert the formula. To do this, i.e., to derive a q-analog of (1), we shall proceed exactly as in the proof of Theorem 7 in [3]. We need a q-analog of the relation

$$\sum_{d=k}^{n} \binom{d-1}{k-1} = \binom{n}{k}$$

which was exploited in [3] in the proof of Theorem 7 as well as in the study of the combinatorial meaning of $R_k(j)$.

The q-binomial coefficient satisfies [2] the recurrence relations

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = q^k \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix},$$

and the second of these gives

$$q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} d \\ k \end{bmatrix} - \begin{bmatrix} d-1 \\ k \end{bmatrix},$$

so that by summing both sides we have the desired q-analog

$$\sum_{d=k}^{n} q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}. \tag{14}$$

We also recall the formula of Meissel [3]

$$\sum_{m \le x} \left[\frac{x}{m} \right] \mu(m) = 1, \tag{15}$$

where μ is the familiar Moebius function in number theory.

We are now in a position to prove

Theorem 2. The bracket function expansion of the q-binomial coefficient is given by

$$(16)\qquad \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{j=k}^{n} \left[\frac{n}{j}\right] R_k(j, q) = \left[\frac{n}{k}\right] + \sum_{j=k+1}^{n} \left[\frac{n}{j}\right] R_k(j, q) ,$$

where

$$(17)\qquad R_k(j, q) = \sum_{\substack{d \mid j \\ d \geq k}} q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} \mu(j/d) .$$

Proof. As in [3, p. 248] we have

$$\sum_{j \leq n} \left[\frac{n}{j}\right] \sum_{d \mid j} q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} \mu(j/d)$$

$$= \sum_{d \leq n} q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} \sum_{m \leq n/d} \left[\frac{n/d}{m}\right] \mu(m)$$

$$= \sum_{d \leq n} q^{d-k} \begin{bmatrix} d-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} ,$$

by (15), then (14).

This completes the proof since it is evident that $R_k(j, q) = 0$ for $j < k$ and $R_k(k, q) = 1$ for all $k \geq 1$ and any q.

We next obtain a Lambert series expansion having $R_k(j, q)$ as coefficient. We need a q-analog of the formula

$$(18)\qquad \sum_{n=k}^{\infty} \binom{n}{k} x^n = x^k (1-x)^{-k-1} , \quad k \geq 0 ,$$

which was used in [3, p. 246].

By using (14), it easily follows that

$$S(k,x) = \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} x^n = \sum_{n=k}^{\infty} x^n \sum_{j=k}^{n} q^{j-k} \begin{bmatrix} j-1 \\ k-1 \end{bmatrix}$$

$$= q^{1-k} x (1-x)^{-1} S(k-1, qx) \quad ,$$

with

$$S(0, qx) = (1 - qx)^{-1} \quad ,$$

so that iteration yields the desired formula

$$(19) \qquad \sum_{n=k}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} x^n = x^k \prod_{j=0}^{k} (1 - q^j x)^{-1}, \quad k \geq 0 .$$

We also recall [3, (3)]

$$(19') \qquad \sum_{n=k}^{\infty} \left[\frac{n}{k} \right] x^n = x^k (1-x)^{-1} (1-x^k)^{-1}, \quad k \geq 1 .$$

We may now state

Theorem 3. The number-theoretic function $R_k(j, q)$ is the coefficient in the Lambert series

$$(20) \qquad \sum_{j=k}^{\infty} R_k(j,q) \frac{x^j}{1-x^j} = x^k \prod_{j=1}^{k} (1 - q^j x)^{-1} = \sum_{n=k}^{\infty} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} q^{n-k} x^n .$$

Indeed, the same steps used in [3, p. 246] apply here. One substitutes in (19) by means of (16), rearranges the series, and then uses (19'). Since we are only concerned with the coefficients in formal generating functions no problem about convergence arises at this point. Later, in Theorem 16, we expand (20) as a power series in a variant form. The right-hand summation in (20) follows easily from (19).

The expansion inverse to (20) is just as easily found, and we state

Theorem 4. The number-theoretic function $A_k(j, q)$ is the coefficient in the expansion

$$\sum_{j=k}^{\infty} A_k(j, q) x^j \prod_{i=1}^{j} (1 - q^i x)^{-1} = \frac{x^k}{1 - x^k} . \tag{21}$$

Indeed, the proof parallels that in [3, 252] in that one starts with (19'), substitutes by means of (12), rearranges, and applies (19).

Now it is evident that the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ is a polynomial of degree $k(n - k)$ in q. Thus it is evident from (4) and (17) that $A_k(j, q)$ and $R_k(j, q)$ are each polynomials in q. In terms of the formal algebra of generating functions we may then equate corresponding coefficients in series to derive identities. Substitution of (20) into (21), and conversely, yields the following orthogonality relations which we state as

Theorem 5. The numbers $A_k(j, q)$ and $R_k(j, q)$ are orthogonal in the sense that

$$\sum_{j=k}^{n} R_k(j, q) A_j(n, q) = \delta_k^n , \tag{22}$$

and

$$\sum_{j=k}^{n} A_k(j, q) R_j(n, q) = \delta_k^n . \tag{23}$$

Thus we have evidently also proved the quite general inversion

Theorem 6. For two sequences $F(n, k, q)$, $G(n, k, q)$, then

$$F(n, k, q) = \sum_{j=k}^{n} G(n, j, q) R_k(j, q) \tag{24}$$

if and only if

$$G(n, k, q) = \sum_{j=k}^{n} F(n, j, q) A_k(j, q) \quad . \tag{25}$$

Again we note that Theorem 6 does not immediately imply Theorem 1 or Theorem 2, as one at least of these must be proved before Theorem 6 yields the other. The expansion and inversion theories are quite separate ideas.

It was seen in [3, p. 247] that the number of compositions of n into k positive summands, $C_k(n)$, is related to $R_k(j)$ by the formula

$$C_k(n) = \binom{n-1}{k-1} = \sum_{d|n} R_k(d) \quad , \tag{26}$$

which was then inverted by the Moebius inversion theorem to get that part of (3) above involving the Moebius function. Since that paper started from the number-theoretic interpretation of $R_k(j)$ and only later used the formula of Meissel to obtain the expansion without starting from the theory of compositions, it is of interest in the present paper to proceed in reverse. The Moebius inversion theorem applied to (17) above gives us at once

Theorem 7. The function $R_k(j, q)$ satisfies the q-analog of (26).

$$q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \sum_{d|n} R_k(d, q) \quad . \tag{27}$$

We now turn to the connections between $R_k(j, q)$ and $A_k(j, q)$ and the Stirling numbers. A formula due to Carlitz was stated in [1] in the form

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{s=k}^{n} \binom{n}{s} (q-1)^{s-k} S_2(k, s-k, q) \quad , \tag{28}$$

where $S_2(n, k, q)$ is a q-Stirling number of the second kind and, explicitly,

$$(29) \qquad S_2(n, k, q) = (q - 1)^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k+n}{k-j} \begin{bmatrix} j+n \\ j \end{bmatrix} .$$

It is evident from the expansions which we have examined here that we may obtain formula (28) in quite a different manner.

Indeed, substitution of (2) into (16) above gives us at once

$$(30) \qquad \begin{bmatrix} n \\ k \end{bmatrix} = \sum_{s=k}^{n} \binom{n}{s} \sum_{j=k}^{s} R_k(j, q) A_j(s) ,$$

and this must agree with (28), so that we are left to assert

Theorem 8. The q-Stirling number of the second kind as defined by (29) may be expressed as

$$(31) \qquad (q - 1)^{s-k} S_2(k, s - k, q) = \sum_{j=k}^{s} R_k(j, q) A_j(s) .$$

This is an interesting result, because when $q = 1$ the left-hand member is zero $(k \neq s)$, and the right-hand member is zero because of the fact of orthogonality of $R_k(j)$ and $A_j(s)$. As a corollary to this theorem we have

Theorem 9. The ordinary Stirling numbers of the second kind (in the author's notation [1]) are given by

$$(32) \qquad S_2(k, n - k) = \lim_{q \to 1} (q - 1)^{k-n} \sum_{j=k}^{n} R_k(j, q) A_j(n) ,$$

where $R_k(j, q)$ is given by (17) and $A_j(n) = A_j(n, 1)$ is given by (4).

It is natural to request a companion formula for the Stirling numbers of the first kind. To attempt this we next need a formula inverse to (28), as the formula inverse to (30) is apparent. We proceed by making use of the q-inversion theorem expressed in relations (10)-(11) above.

Put

$$\binom{n}{k} = \sum_{s=0}^{n} \binom{n}{s} f(s, k, q) . \tag{33}$$

then by (10)-(11) this inverts to yield

$$f(n, k, q) = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} \binom{j}{k} \tag{34}$$

It was found in [1, (3.19)] that the q-Stirling numbers of the first kind as there defined could be expressed in the form

$$S_1(n, k, q) = (q - 1)^{-k} \sum_{j=0}^{k} (-1)^{k-j} \binom{n-j}{k-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)/2} , \tag{35}$$

which may be rewritten as follows:

$$\begin{aligned} S_1(n, n-k, q) &= (q-1)^{k-n} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-j}{n-k-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)/2} , \\ &= (q-1)^{k-n} \sum_{j=0}^{n} (-1)^{n-k-j} \binom{n-j}{k} \begin{bmatrix} n \\ j \end{bmatrix} q^{j(j+1)/2} , \\ &= (q-1)^{k-n} \sum_{j=0}^{n} (-1)^{k-j} \binom{j}{k} \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j+1)/2} , \end{aligned}$$

so that we may write

$$S_1(n, n-k, q) = (1-q)^{k-n} \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} \binom{j}{k} q^{(n-j)(n-j-1)/2} q^{n-j} . \tag{36}$$

This looks somewhat like $f(n, k, q)$ as given by (34), but with an important difference: the factor q^{n-j}. It seems rather difficult to modify the work so as to remove this factor and express $f(n, k, q)$ easily in terms of $S_1(n, k, q)$. We could call $f(n, k, q)$ a modified Stirling number of the first kind. We illustrate further the difficulty involved. Instead of (33) let us put

$$q^{-n} \binom{n}{k} = \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix} g(s, k, q) . \tag{37}$$

This inverts by (10)-(11) to give

$$g(n, k, q) = \sum_{j=0}^{n} (-1)^{n-j} \begin{bmatrix} n \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} \binom{j}{k} q^{-j} ,$$

and comparison of this with (36) yields at once

$$g(n, k, q) = q^{-n} (1-q)^{n-k} S_1(n, n-k, q) . \tag{38}$$

This, however, leads to difficulty when we examine the analog of (30). Indeed, substitution of (12) into (1) gives us at once

$$\binom{n}{k} = \sum_{s=k}^{n} \begin{bmatrix} n \\ s \end{bmatrix} \sum_{j=k}^{s} R_k(j) A_j(s, q) . \tag{39}$$

However, expansion (37) gives us

$$\binom{n}{k} = \sum_{s=0}^{n} \begin{bmatrix} n \\ s \end{bmatrix} q^n g(s, k, q) , \tag{40}$$

and we may not equate coefficients since (39) requires the coefficient of the q-binomial coefficient to be independent of n, but in (40) it is not.

Of course, by (33) and (39) we do have

$$\sum_{j=k}^{s} R_k(j)A_j(s,q) = \sum_{j=k}^{s}(-1)^{s-j}\begin{bmatrix} s \\ j \end{bmatrix}\binom{j}{k} q^{(s-j)(s-j-1)/2}, \tag{41}$$

which is the best companion to (31) noted at this time.

Another approach would be to develop a q-bracket function (q-greatest integer function) and proceed in a manner similar to the above by expanding the binomial coefficient $\binom{n}{j}$ in terms of a q-bracket function and using this in relation (2) just as we here used relation (2) in (16) to get (30) and then (31). The development of the q-analog of the greatest integer function will be left for a separate account.

It seems not without interest to exhibit a numerical example of (32). From definition, $S_2(2,3) = 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 2 = 15$, being the sum of the 4 possible products, each with 3 factors (repetition allowed), which may be formed from the first 2 natural numbers. The table of values of $A_j(n)$ in [3, p. 254] and the formula (17) may be used. We find that

$$\begin{aligned} S_2(2,3) = S_2(2, 5-2) &= \lim_{q\to 1} (q-1)^{-3}\sum_{j=2}^{5} R_2(j,q)A_j(5) \\ &= \lim_{q\to 1}(q-1)^{-3}(-8+6q(q+1)-4(-1+q^2+q^3+q^4)+(q^3+q^4+q^5+q^6)) \\ &= \lim_{q\to 1}(q-1)^{-3}(-4+6q+2q^2-3q^3-3q^4-q^5+q^6) = 15, \end{aligned}$$

the limit being easily found by l'Hospital's theorem.

We should remark for the convenience of the reader that the Stirling numbers appear in various forms of notation and the notations of Riordan [5], Jordan [4], and the author [1] are related as follows:

$$s(n,k) = S_n^k = (-1)^{n-k}S_1(n-1, n-k), \tag{42}$$

and

$$S(n, k) = \mathfrak{S}_n^k = S_2(k, n - k) = \frac{1}{k!} \Delta^k 0^n . \tag{43}$$

The S_1 and S_2 notations are convenient because of the generating functions

$$\prod_{k=0}^{n} (1 + kx) = \sum_{k=0}^{n} S_1(n, k)x^k, \quad \prod_{k=0}^{n} (1 - kx)^{-1} = \sum_{k=0}^{\infty} S_2(n, k)x^k . \tag{44}$$

Also, in [1] will be found a discussion of the interesting continuation formulas

$$S_2(-n-1, k, 1/q) = q^k S_1(n, k, q), \quad S_1(-n-1, k, 1/q) = q^k S_2(n, k, q) . \tag{45}$$

A q-polynomial was suggested in [1] which would include both S_1 and S_2 as instances. The q-Stirling numbers as defined in [1] satisfy the generating relations

$$\prod_{k=0}^{n} (1 + [k]x) = \sum_{k=0}^{n} S_1(n, k, q)x^k, \quad \prod_{k=0}^{n} (1 - [k]x)^{-1} = \sum_{k=0}^{\infty} S_2(n, k, q)x^k, \tag{46}$$

in analogy to (44). Here $[k]$ is called a q-number and is defined by

$$\left[k\right]_q = \left[k\right] = \frac{q^k - 1}{q - 1} ,$$

so that

$$\lim_{q \to 1} \left[k\right] = k .$$

The notation $[k]$ must not be confused with that for the bracket function.

Relations (31) and (41) suggest that we consider the following. By using Theorem 3 with base q, and substituting with Theorem 4 and base p, we find the identity

$$(47) \qquad x^k \prod_{j=1}^{k} (1 - q^j x)^{-1} = \sum_{n=k}^{\infty} x^n \prod_{i=1}^{n} (1 - p^i x)^{-1} \sum_{j=k}^{n} R_k(j, q) A_j(n, p) \ .$$

It will be recalled from [3] that for $p = q$ the inner sum is merely a Kronecker delta. In view of Theorem 8, we may look on the sum

$$(48) \qquad \sum_{j=k}^{n} R_k(j, q) A_j(n, p) = f(n, k, p, q)$$

as a kind of generalized Stirling number.

Some of the results already found extend to real numbers instead of natural numbers only. The product definition (9) holds for $n = x =$ real number. We may also extend the range of validity of (16) just as was done in the proof of Theorem 7 in [3]. Indeed we have

Theorem 10. For two sequences $F(x, k, q)$, $G(x, k, q)$, then for real x and all natural numbers k

$$(49) \qquad F(x, k, q) = \sum_{k \le j \le x} G(x, j, q) R_k(j, q)$$

if and only if

$$(50) \qquad G(x, k, q) = \sum_{k \le j \le x} G(x, j, q) A_k(j, q) \ ,$$

where R and A are defined by (17) and (13).

The proof uses nothing more than Theorem 5.

The real-number extension of Theorem 1 most readily found is as follows.

Theorem 11. For real x and natural numbers k

$$(51) \qquad \begin{bmatrix} x \\ k \end{bmatrix} = \sum_{k \le j \le x} \begin{bmatrix} [x] \\ j \end{bmatrix}_q A_k(j, q) \ .$$

The proof parallels that of Theorem 7 in [3]. Note that the 'expansion'

$$\left[\frac{x}{k}\right] = \sum_{k \leqq j \leqq x} \begin{bmatrix} x \\ j \end{bmatrix}_q A_k(j, q) \tag{52}$$

is <u>incorrect.</u> What is really expanded in (51) is

$$\left[\frac{[x]}{k}\right], \text{ however in fact } \left[\frac{[x]}{k}\right] = \left[\frac{x}{k}\right],$$

so that what one might first try from (50) does not hold.

Similarly, a correct generalization of Theorem 2, by inversion of (51), is

<u>Theorem 12.</u> For real x and natural numbers k

$$\begin{bmatrix} [x] \\ k \end{bmatrix}_q = \sum_{k \leqq j \leqq x} \left[\frac{x}{j}\right] R_k(j, q) . \tag{53}$$

The failure of (52) suggests two new procedures. First, we may define a kind of <u>q-greatest integer function</u> (not the only possible definition) by

$$\left[\frac{x}{k}, q\right] = \sum_{k \leqq j \leqq x} \begin{bmatrix} x \\ j \end{bmatrix}_q A_k(j, q) , \tag{54}$$

and secondly, we may introduce new coefficients such that

$$\left[\frac{x}{k}\right] = \sum_{k \leqq j \leqq x} \begin{bmatrix} x \\ j \end{bmatrix}_q B_k(j, q) , \tag{55}$$

but these are not easily determined. We shall leave a detailed discussion of such extensions for another paper.

Although we omit a detailed study of the arithmetical properties of the functions $R_k(j,q)$ and $A_k(j,q)$, we remark that such a study makes use of arithmetical properties of the q-binomial coefficients. Fray [6] has recently announced some results in that direction. In particular he announces the following theorem. Let q be rational and $q \not\equiv 0 \pmod{p}$, and let e = exponent to which q belongs (mod p). Let $n = a_0 + ea$, $0 \le a_0 < e$, and $k = b_0 + eb$, $0 \le b_0 < e$. Then

$$\begin{bmatrix} n \\ k \end{bmatrix} \equiv \begin{bmatrix} a_0 \\ b_0 \end{bmatrix} \binom{a}{b} \pmod{p} \quad . \tag{56}$$

We do explore certain arithmetical properties which are of a different nature. First of all, (17) gives

$$qR_1(n,q) = \sum_{d|n} q^d \mu(n/d) \quad ,$$

and by a theorem of Gegenbauer [3, p. 256] this sum is always divisible by n for any natural number q. Thus we have the congruence

$$qR_1(n,q) \equiv 0 \pmod{n} \tag{57}$$

for all integers n, q. This is trivial for $R_1(n,1) = R_1(n) = 0$ for $n \ge 2$.

On the other hand, let n = p be a prime. Then we have for integers q

$$R_1(p,q) = q^{p-1} - 1 \equiv 0 \pmod{p}, \quad \text{for } (p,q) = 1 \; , \tag{58}$$

this following from the Fermat congruence. Again this is trivial when q = 1.

It is possible to obtain various identical congruences for the functions studied in this paper. If f(q) and g(q) are two polynomials in q with integer coefficients, we recall that $f(q) \equiv g(q) \pmod{m}$ is an identical congruence (mod m) provided that respective coefficients of powers of q are congruent. We shall call such congruences <u>identical q-congruences</u>. Thus we have

Theorem 13. The functions defined by (13) and (34) satisfy the identical q-congruence

$$A_k(n, q) \equiv f(n, k, q) \pmod{k} \quad (k \geq 2,\ n = 1, 2, 3, \cdots)$$

if and only if k is prime.

Proof. Apply (8) to (13) and (34).

Another way of seeing this is to note that (33) and (39) imply

$$f(n, k, q) = \sum_{j=k}^{n} R_k(j) A_j(n, q) = A_k(n, q) + \sum_{j=k+1}^{n} R_k(j) A_j(n, q) ,$$

and recall (7), whence the result follows.

In similar fashion one can obtain various congruences involving the q-Stirling numbers.

As a final remark about identical congruences we wish to note the following q-criterion for a prime.

Theorem 14. The identical q-congruence (for $k \geq 2$)

$$(59) \qquad (1 - q)^{k-1} \equiv [k]_q \pmod{k}$$

is true if and only if k is a prime. Here, the q-number

$$[k]_q = (q^k - 1)/(q - 1) .$$

Proof. We shall use the easily established q-analog identity:

$$(60) \qquad (q - 1)^{k-1} = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} [j]_q .$$

From this we have

$$(61) \qquad (q - 1)^{k-1} - [k]_q = \sum_{j=1}^{k-1} (-1)^{k-j} \binom{k}{j} [j]_q .$$

Now it is easily seen that

$$k \,\Big|\, \binom{k}{j} \quad \text{if } k = \text{prime and } 1 \le j \le k - 1 .$$

Hence it is trivial that (59) holds when $k = \text{prime}$.

Assume then that (59) holds for a composite k. Then we have (61) so that

$$k \,\Big|\, \binom{k}{j}, \quad 1 \le j \le k - 1 .$$

Let p be a prime divisor of k. Then for some value of j, $1 \le j \le k - 1$, $j = p$, whence $k \,\Big|\, \binom{k}{p}$. Considering this in the form

$$k \,\Big|\, \frac{k(k-1)\cdots(k-p+1)}{p(p-1)!}$$

we have $(k, j) = 1$, whence k is relatively prime to every factor $k - j$ in the numerator and we have $p(p-1)! \,\big|\, (k-1)(k-2)\cdots(k-p+1)$. This implies that $p \,|\, (k - j)$ for some j with $1 \leq j \leq k - 1$, or since $p \,|\, k$ (by hypothesis), therefore $p \,|\, j$ which is impossible. Thus the only possibility is that k is prime itself.

If we write out the congruence as

$$(1-q)^{k-1} \equiv \frac{1-q^k}{1-q} \pmod{k} ,$$

and multiply through by $1 - q$ we have the equivalent identical congruence

$$(1-q)^k \equiv 1 - q^k \pmod{k} \tag{62}$$

if and only if $k = \text{prime}$ $(k \ge 2)$.

It was noted in [3] that E. M. Wright's proof of (8) was to show that (8) is equivalent to the identical q-congruence (62). We note a typographical mistake in [3, p. 241] in that the identical congruence there should read

$$(1 - x)^p \equiv 1 - x^p \pmod{p} \tag{63}$$

if and only if p is prime.

The proof above for (59) is equivalent to Wright's proof of (62), however it is felt to be of interest to present it by way of the q-identity (61). Of course, the generating functions (1) and (2) show that (8) and (63) are equivalent.

Since [3] was concerned with compositions and partitions, it is of interest to recall a theorem of Cayley to the effect that the number of partitions of n into j or fewer parts, each summand $\leq i$, is the coefficient of q^n in the series expansion of the q-binomial coefficient

$$\begin{bmatrix} j+i \\ j \end{bmatrix} = \prod_{k=1}^{j} \frac{1 - q^{k+i}}{1 - q^k} .$$

When $|q| < 1$ and $i \to \infty$, $j \to \infty$, this reduces to Euler's formula for the partition of n into any number of parts at all,

$$\prod_{k=1}^{\infty} (1 - q^k)^{-1} = 1 + \sum_{n=1}^{\infty} p(n)\, q^n .$$

It is expected that the q-identities derived here have further implications for partitions and compositions.

As another result we show that $A_k(n, q)$ may be written in such a way that the greatest integer function does not explicitly appear. This is analogous to relation (41) in [3]. We have

Theorem 15. For the numbers defined by (13) we have

$$A_k(n, q) = \sum_{1 \leq m \leq n/k} (-1)^{n-mk} \begin{bmatrix} n-1 \\ mk-1 \end{bmatrix} q^{(n-mk)(n-mk+1)/2} . \tag{64}$$

Proof. Recall that

$$\begin{bmatrix} n \\ j \end{bmatrix} = \begin{bmatrix} n-1 \\ j \end{bmatrix} + \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{n-j} .$$

Then by (13) and this we have

$$A_k(n, q) = \sum_{j=0}^{n-1} (-1)^{n-j} \begin{bmatrix} n-1 \\ j \end{bmatrix} q^{(n-j)(n-j-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}$$

$$+ \sum_{j=1}^{n} (-1)^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{n-j} q^{(n-j)(n-j-1)/2} \begin{bmatrix} j \\ k \end{bmatrix}$$

$$= \sum_{j=1}^{n} (-1)^{n-j+1} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{(n-j)(n-j+1)/2} \begin{bmatrix} j-1 \\ k \end{bmatrix}$$

$$+ \sum_{j=1}^{n} (-1)^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{(n-j)(n-j+1)/2} \begin{bmatrix} j \\ k \end{bmatrix}$$

$$= \sum_{j=1}^{n} (-1)^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{(n-j)(n-j+1)/2} \left\{ \begin{bmatrix} j \\ k \end{bmatrix} - \begin{bmatrix} j-1 \\ k \end{bmatrix} \right\}$$

$$= \sum_{\substack{k \le j \le n \\ k \mid j}} (-1)^{n-j} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix} q^{(n-j)(n-j+1)/2} ,$$

which may then be written as we indicate, letting $j = mk$ in the summation.

An alternative form of the power series expansion for (20) is easily found. Indeed, the product on the right side of (20) may be written as follows:

$$\prod_{j=1}^{k} (1 - q^j x)^{-1} = \prod_{j=0}^{k-1} (1 - q^j q x)^{-1} = \prod_{j=0}^{\infty} \frac{1 - xq q^{j+k}}{1 - xq q^j} .$$

However, Carlitz [7, p. 525] has noted the expansion (due to Cauchy [8])

$$\prod_{j=0}^{\infty} \frac{1 - atq^j}{1 - btq^j} = \sum_{n=0}^{\infty} \frac{(b - a)_n}{(q)_n} t^n ,$$

where

$$(b - a)_n = \prod_{j=0}^{n-1} (b - q^j a) ,$$

and

$$(q)_n = \prod_{j=1}^{n} (1 - q^j) .$$

Setting $a = q^k$, $b = 1$, $t = qx$, we can obtain the desired expansion. We state the result as

Theorem 16. The Lambert series for $R_k(j, q)$ may be written as a power series in the form

(65)
$$\sum_{j=k}^{\infty} R_k(j, q) \frac{x^j}{1 - x^j} = \sum_{n=0}^{\infty} \frac{(1 - q^k)_n}{(q)_n} q^n x^{n+k} .$$

Further results relating to compositions and partitions will be left for a future paper.

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