ADVANCED PROBLEMS AND SOLUTIONS

Edited By

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-126 Proposed by L. Carlitz, Duke University.

Let F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively. Sum the series

$$\sum_{m,n=0}^{\infty} F_m F_n F_{m+n} x^m y^n, \qquad \sum_{m,n=0}^{\infty} F_m F_n L_{m+n} x^m y^n .$$

Sum the series

$$\sum_{\substack{m,n=0 \\ m+n \text{ even}}}^{\infty} F_m F_n x^m y^n, \qquad \sum_{\substack{m,n=0 \\ m+n \text{ even}}}^{\infty} L_m L_n x^m y^n .$$

Sum the series

$$S = \sum_{m,n,p=0}^{\infty} F_{n+p} F_{p+m} F_{m+n} x^m y^n z^p .$$

H-127 Proposed by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.

The Fibonacci polynomials are defined by

$$f_{n+1}(x) = x \cdot f_n(x) + f_{n-1}(x) \qquad (n \geq 2)$$

$$f_1(x) = 1 \quad \text{and} \quad f_2(x) = x \; .$$

If $z_r = f_r(x) \cdot f_r(y)$, then show that

(i) z_r satisfies the recurrence relation

$$z_{n+4} - xy \cdot z_{n+3} - (x^2 + y^2 + 2)\, z_{n+2} - xy \cdot z_{n+1} + z_n = 0 \cdot$$

(ii) $$(x + y)^2 \sum_1^n z_r = (z_{n+2} - z_{n-1}) - (xy - 1)(z_{n+1} - z_n) \; .$$

H-128 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let F_n and L_n denote the Fibonacci and Lucas numbers, respectively. Show that

$$F_n \equiv 2^{2n+3} - 2^{3n+3} \pmod{11} \; ,$$

$$L_n \equiv 2^{2n} + 2^{3n} \pmod{11} \; .$$

Generalize.

H-129 Proposed by Stanley Rabinowitz, Far Rockaway, New York.

Define the Fibonacci polynomials by $f_1(x) = 1$,

$$f_1(x) = 1, \;\; f_2(x) = x, \;\; f_{n+2}(x) = xf_{n+1}(x) + f_n(x), \quad n > 0 \; .$$

Solve the equation

$$(x^2 + 4)f_n^2(x) = 4k(-1)^{n-1}$$

in terms of radicals, where k is a constant.

SOLUTIONS

GREATEST POWER OF TWO IN N

H-81 Proposed by Vassili Daiev, Sea Cliff, New York.

Find the n^{th} term of the sequence

1, 1, 3, 1, 5, 3, 7, 1, 9, 5, 11, 3, 13, 7, 15, 1, 17, 9, 19, 5, ···

Solution by J. L. Brown, Jr., Ordnance Research Laboratory, State College, Pa.

Let u_n denote the n^{th} term of the sequence for $n \geq 1$. Then for $n \geq 1$, each integer n has a unique representation in the form

$$n = 2^{k(n)} \cdot r(n) \quad ,$$

where k(n) is a non-negative integer and r(n) is an odd integer ≥1. The given sequence is formed by the rule $u_n = r(n)$.

Also solved by Thomas Dence, L. Carlitz, and C. B. A. Peck.

LEHMER'S FAMOUS PROBLEM GENERALIZED

H-82 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

If $f_0(x) \equiv 0$ and $f_1(x) \equiv 1$, $f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$, then show

$$\tan^{-1}\frac{1}{x} = \sum_{n=1}^{\infty} \tan^{-1}\left(\frac{x}{f_{2n+1}(x)}\right)$$

Solution by M.N.S. Swamy, Nova Scotia Technical College, Halifax, Canada.

Let $\tan \theta_n = 1/f_n(x)$. Then

$$\tan(\theta_{2n} - \theta_{2n+2}) = \frac{f_{2n+2}(x) - f_{2n}(x)}{1 + f_{2n}(x)f_{2n+2}(x)} = \frac{xf_{2n+1}(x)}{1 + f_{2n}(x)f_{2n+2}(x)}$$

It may be easily established by induction that,

$$f_{n-1}(x)\, f_{n+1}(x) - f_n^2(x) = (-1)^n$$

Hence,

$$\tan(\theta_{2n} - \theta_{2n+2}) = \frac{xf_{2n+1}(x)}{f_{2n+1}^2(x)} = \frac{x}{f_{2n+1}(x)}$$

Or,

$$\tan^{-1}[1/f_{2n}(x)] - \tan^{-1}[1/f_{2n+2}(x)] = \tan^{-1}[x/f_{2n+1}(x)]$$

Hence,

$$\sum_1^m \tan^{-1}\left\{\frac{x}{f_{2m+1}(x)}\right\} = \tan^{-1}\left\{\frac{1}{f_2(x)}\right\} - \tan^{-1}\left\{\frac{x}{f_{2m+2}(x)}\right\}$$

Now $f_2(x) = x$. Also, as $m \to \infty$, $\tan^{-1}(1/f_{2m+2}(x)) \to 0$. Hence,

$$\sum_1^\infty \tan^{-1}\frac{x}{f_{2n+1}(x)} = \tan^{-1}\frac{1}{x}$$

Note: Since $f_n(1) = F_n$, the n^{th} Fibonacci number, we get the interesting result that,

$$\tan^{-1} 1 = \frac{\pi}{4} = \sum_{1}^{\infty} \tan^{-1} \frac{1}{F_{2n+1}}$$

This is Lehmer's famous result.

Also solved by Joseph D. E. Konhauser.

ANOTHER CUTIE

H-83 Proposed by Mrs. William Squire, Morgantown, West Virginia.

Show

$$\sum_{t=1}^{\left[\frac{m+1}{2}\right]} (-1)^{t-1} \binom{m-t}{t-1} 3^{m+1-2t} = F_{2m} ,$$

where $[x]$ is the greatest integer function.

Solution by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

We know that the Chebyshev polynomial $S_n(x)$ is given by

$$S_n(x) = \sum_{j=0}^{[n/2]} (-1)^j \binom{n-j}{j} x^{n-2j} \tag{1}$$

Also, $S_n(x)$ satisfies the difference equation

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x) ,$$

with

$$S_0 = 1, \qquad S_1 = x .$$

Hence,

$$S_n(x) = \frac{1}{\sqrt{x^2 - 4}}\left[\left\{\frac{x + \sqrt{x^2 - 4}}{2}\right\}^{n+1} - \left\{\frac{x - \sqrt{x^2 - 4}}{2}\right\}^{n+1}\right]$$

Or,

$$S_n(3) = \frac{1}{\sqrt{5}}\left\{\left(\frac{3 + \sqrt{5}}{2}\right)^{n+1} - \left(\frac{3 - \sqrt{5}}{2}\right)^{n+1}\right\}$$

$$= \frac{1}{\sqrt{5}}\left\{\left(\frac{1 + \sqrt{5}}{2}\right)^{2n+2} - \left(\frac{1 - \sqrt{5}}{2}\right)^{2n+2}\right\} = F_{2n+2}$$

Hence, $F_{2n} = S_{n-1}(3)$. Therefore, from (1) we get

$$F_{2m} = \sum_{j=0}^{[(m-1)/2]} (-1)^j \binom{m - j - 1}{j} 3^{m-1-2j}$$

Changing j to $(t - 1)$ we have,

$$\sum_{t=1}^{[(m+1)/2]} (-1)^{t-1} \binom{m - t}{t - 1} 3^{m+1-2t} = F_{2m}$$

ALPHA AND BETA, AGAIN!

H-85 Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.

Let

$$D_n = f_n x^n - [f_n x^n],$$

where

$$f_{n+1} = f_n + f_{n-1}$$

with

$$f_0 = f_1 = 1, \qquad x = (1 + \sqrt{5})/2,$$

and $[z]$ = greatest integer $\leq z$ (so that $z - [z]$ = fractional part of z). Prove (or disprove) the existence of the limits

$$\lim_{n\to\infty} D_{2n} = 0.27\cdots = A$$

and

$$\lim_{n\to\infty} D_{2n+1} = 0.72\cdots = B \text{ with } A + B = 1.$$

Generalize to case of

$$\mu_{n+1} = p\mu_n + q\mu_{n-1},$$

where p and q are real and μ_0 and μ_1 are given.

Solution by L. Carlitz, Duke University.

Put

$$x = \frac{1}{2}(1 + \sqrt{5}), \quad y = \frac{1}{2}(1 - \sqrt{5}),$$

so that

$$f_n = \frac{x^{n+1} - y^{n+1}}{x - y}, \quad D_n = \frac{x^{2n+1} - x^n y^{n+1}}{x - y} - \left[\frac{x^{2n+1} - x^n y^{n+1}}{x - y}\right].$$

Then

$$D_{2n} = \frac{x^{4n+1} - y}{x - y} - \left[\frac{x^{4n+1} - y}{x - y}\right],$$

$$\frac{x^{4n+1} - y}{x - y} = F_{4n} + \frac{y^{4n+1} - y}{x - y} = F_{4n} + \frac{y^2 - y^{4n+2}}{y^2 + 1}$$

Since $-1 < y < 0$ it follows that

$$0 < \frac{y^2 - y^{4n+2}}{y^2 - 1} < 1 \qquad (n > 0) \;.$$

Thus

$$\left[\frac{x^{4n+1} - y}{x - y}\right] = F_{4n} \;, \quad D_{2n} = \frac{y^2 - y^{4n+2}}{y^2 + 1} \quad .$$

Therefore

$$\lim_{n=\infty} D_{2n} = \frac{y^2}{y^2 + 1} = \frac{1}{x^2 + 1} = \frac{2}{5 + \sqrt{5}} = \frac{5 - \sqrt{5}}{10} = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5} - 1}{2}\right)$$

Similarly

$$D_{2n+1} = \frac{x^{4n+3} + y}{x - y} - \left[\frac{x^{4n+3} + y}{x - y}\right] \;,$$

$$\frac{x^{4n+3} + y}{x - y} = F_{4n+2} + \frac{y^{4n+3} + y}{x - y} = F_{4n+2} - \frac{y^2 + y^{4n+4}}{y^2 + 1} \quad .$$

Since

$$0 < \frac{y^2 + y^{4n+4}}{y^2 + 1} < 1 \;,$$

we have

$$\left[\frac{x^{4n+3} + y}{x - y}\right] = F_{4n+2} \qquad , \quad D_{2n+1} = 1 - \frac{y^2 + y^{4n+4}}{y^2 + 1} \quad ,$$

$$\lim_{n=\infty} D_{2n+1} = 1 - \frac{y^2}{y^2 + 1} = \frac{x^2}{x^2 + 1} = \frac{5 + \sqrt{5}}{10} = \frac{1}{\sqrt{5}}\left(\frac{1 + \sqrt{5}}{2}\right) \;.$$

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