ON A PARTITION OF GENERALIZED FIBONACCI NUMBERS

S. G. Mohanty
McMaster University,
Hamilton, Ontario, Canada

As a continuation of results in [4], this paper deals with the concept of minimal and maximal representations of positive integers as sums of generalized Fibonacci numbers (G. F. N.) defined below and presents a partition of the G. F. N. in relation to either minimal or maximal representation.

Consider the sequence $\{F_t\}$, where

$$(1) \qquad F_1 = F_2 = \cdots = F_r = 1, \quad r \geq 2 \quad \text{and} \quad F_t = F_{t-1} + F_{t-r}, \quad t \geq r .$$

Obviously, the sequence gives rise to the sequence of Fibonacci numbers for $r = 2$. For this reason, we call $\{F_t\}$ a sequence of G. F. N. Clearly $\{F_t\}$ is a special case of Daykin's Fibonacci sequence [3], as well as of Harris and Styles' sequence [6].

We remark that it is possible to express any positive integer N as a sum of distinct F_i's, subject to the condition that $F_1, F_2, \cdots, F_{r-1}$ are not used in any sum (reference: Daykin's paper [3]). In other words, we can have

$$(2) \qquad N = \sum_{i=r}^{s} a_i F_i$$

with $a_s = 1$ and $a_i = 1$ or 0, $r \leq i < s$. Here s is the largest integer such that F_s is involved in the sum.

<u>Definition 1:</u> In case (2) is satisfied, the vector $(a_r, a_{r+1}, \cdots, a_s)$ of elements 1 or 0 with $a_s = 1$, is called a representation of N in $\{F_t\}$, having its index as s.

<u>Definition 2:</u> A representation of N in $\{F_t\}$ is said to be minimal or maximal according as $a_i a_{i+j} = 0$ or $a_i + a_{i+j} \geq 1$, for all $i \geq r$ and $j = 1, 2, \cdots, r - 1$.

Definition 2 is just an extension of that by Ferns [4] to $r > 2$.

Now, we state some results in the forms of lemmas, to be used subsequently.

Lemma 1:

(i) Every positive integer N has a unique minimal representation;

(ii) The index of the minimal representation of N ($F_n \le N < F_{n+1}$, $n \ge r$) is n. If $N = F_n$, then $a_r = a_{r+1} = \cdots = a_{n-1} = 0$.

Lemma 2:

(i) Every positive integer N has a unique maximal representation:

(ii) The index of the maximal representation of N ($F_{n+r-1} - r < N \le F_{n+r} - r$, $n \ge r$) is n. If $N = F_{n+r} - r$, then $a_r = a_{r+1} = \cdots = a_{n-1} = 1$.

Lemma 3. If the minimal representation of

$$N' = F_{n+r} - r - N \qquad (F_u \le N' < F_{u+1},\ n \ge r,\ r \le u \le n-1)$$

is

$$(a_r,\ a_{r+1},\ \cdots,\ a_u),$$

then the maximal representation of N

$$(F_{n+r} - F_{u+1} - r < N \le F_{n+r} - F_u - r)$$

is

$$(1 - a_r,\ 1 - a_{r+1},\ \cdots,\ 1 - a_u,\ \underbrace{1,\ 1,\ \cdots,\ 1}_{n-u})$$

and conversely.

Note that Lemma 3 provides a method of construction of the maximal (minimal) representation, given the minimal (maximal) representation of integers. Furthermore, the last time a zero occurs in the maximal representation of N ($F_{n+r} - F_{u+1} - r < N \le F_{n+r} - F_u - r$), is at the $(u - r + 1)$st position, that is, $1 - a_u = 0$.

Proof. The proof of Lemma 1 is given in [3] as Theorem C. (Also see Brown's paper [1].) A generalized argument similar to that in the proof of

Theorem 1 in [2] would lead us to Lemma 2 and Lemma 3. However, the basic steps of the proof are indicated.

First, we assert that

$$\sum_{i=r}^{n} F_i + r = F_{r+n} \quad . \tag{3}$$

When $n \geq 2r - 1$,

$$\sum_{i=r}^{n} F_i + r = F_r + \cdots + F_{2r-2} + F_{2r-1} + F_{2r} + \cdots + F_n$$

(since $r = F_{2r-1}$)

$$= F_{r+1} + \cdots + F_{2r-1} + 2F_{2r} + F_{2r+1} + \cdots + F_n$$

$$\cdots$$

$$= F_{r+n} \, .$$

When $n < 2r - 1$, (3) can also be checked.

Next, we develop a method to construct a maximal representation from the system of minimal representation, and finally show that this representation is unique.

When

$$F_{n+r-1} - r < N \leq F_{n+r} - r \; ,$$

i. e.,

$$\sum_{i=r}^{n-1} F_i < N \leq \sum_{i=r}^{n} F_i, \quad \text{(by (3))}$$

we get

$$N' = F_{n+r} - r - N = \sum_{i=r}^{n} F_i - N < \sum_{i=r}^{n} F_i - \sum_{i=1}^{n-1} F_i = F_n \,. \tag{4}$$

Because of (4) and Lemma 1, let us assume that

$$F_u \leq N' < F_{u+1}, \quad r \leq u \leq n - 1 \;,$$

and that N' has the minimal representation $(a_r, a_{r+1}, \cdots, a_u)$. Thus, $(b_r, b_{r+1}, \cdots, b_n)$, where

$$b_i = \begin{cases} 1 - a_i \;, & i = r,\; r+1, \cdots, u, \\ 1 \;, & i = u+1,\; u+2, \cdots, n, \end{cases}$$

is a maximal representation of N as we can show that $b_i + b_{i+j} \geq 1$ from $a_i a_{i+j} = 0$ for all $i \geq r$ and $j = 1, 2, \cdots, r - 1$.

Suppose that two maximal representations of N are given by

$$N = \sum_{i=r}^{n} a_i F_i = \sum_{i=r}^{n} a'_i F_i \;, \quad a_n = a'_{n'} = 1 \;,$$

with $n > n'$. Letting $n = cr + d$, we obtain

$$\begin{aligned} \sum_{i=r}^{n} a_i F_i \geq F_n &+ F_{n-2} + F_{n-3} + \cdots + F_{n-r} \\ &+ F_{n-r-2} + F_{n-r-3} + \cdots + F_{n-2r} \\ &+ \cdots \\ &+ F_{n-(c-2)r-2} + \cdots + F_{n-(c-1)r} \end{aligned} \tag{5}$$

$$= F_{n+1} + F_{n-1} + F_{n-2} + \cdots + F_{n+2-r} - (r - 1)$$

$$= F_{n+r-1} - (r - 1) \,.$$

But

$$\sum_{i=r}^{n'} a'_i F_i \le \sum_{i=r}^{n'} F_i \le \sum_{i=r}^{n-1} F_i = F_{n+r-1} - r \text{, by (3)} .$$

This is a contradiction of (5) and therefore $n = n'$.

From

$$N = \sum_{i=r}^{n} (1 - a_i)F_i = \sum_{i=r}^{n} (1 - a'_i)F_i$$

it follows that

$$N^* = \sum_{i=r}^{n} (1 - a_i)F_i = \sum_{i=r}^{n} (1 - a'_i)F_i$$

which corresponds to two admissible minimal representations of N^*. The proof is complete, due to Lemma 1.

Definition 3: Define $U(n; m_1, m_2, \cdots, m_r)$ as the number of positive integers N satisfying the following: (the definition arises as a natural consequence of Lemma 1)

(i) $F_n \le N < F_{n+1}$, $n \ge r$;

(ii) In the minimal representation $(a_r, a_{r+1}, \cdots, a_n)$ of N, there are exactly $m_i a'_\alpha$'s among non-zero a's except a_n, such that $\alpha \equiv i - 1 \pmod r$ $i = 1, 2, \cdots, r$.

An illustration of the definition for $r = 3$, might serve a useful purpose. Consider all integers N, $F_{10} \le N < F_{11}$ and their respective minimal representations are:

$19 = F_{10}$, $20 = F_3 + F_{10}$, $21 = F_4 + F_{10}$, $22 = F_5 + F_{10}$, $23 = F_6 + F_{10}$,
$24 = F_3 + F_6 + F_{10}$, $25 = F_7 + F_{10}$, $26 = F_3 + F_7 + F_{10}$, $27 = F_4 + F_7 + F_{10}$.

Then we have

$$U(10;\ 0,\ 0,\ 0) = 1,\ \ U(10,\ 1,\ 0,\ 0) = 2,\ \ U(10;\ 0,\ 1,\ 0) = 2,$$
$$U(10;\ 0,\ 0,\ 1) = 1,\ \ U(10;\ 2,\ 0,\ 0) = 1,\ \ U(10,\ 1,\ 1,\ 0) = 1,$$
$$U(10,\ 0,\ 2,\ 0) = 1\ .$$

It may be observed that a_n is omitted in the definition without any ambiguity, as it is present in every representation. Furthermore, it is significant to note that Definition 3 gives rise to a partition of the G. F. N.

Following the procedure in [4] on pages 23 and 24, we can show that either by replacing F_{n-1} by F_n in the minimal representation of every N_1, $F_{n-1} \le N_1 < F_n$, or by adding F_n in the minimal representation of every N_2, $F_{n-r} \le N_2 < F_{n-r+1}$, we get the minimal representation of every N, $F_n \le N < F_{n+1}$. Therefore, $U(n; m_1, m_2, \cdots, m_r)$ satisfies the following difference equations:

For $m > 1$,

$$(6)\quad \begin{cases} U(rm; m_1, m_2, \cdots, m_r) = U(rm-1; m_1, m_2, \cdots, m_r) \\ \qquad\qquad + U(r(m-1); m_1 - 1, m_2, \cdots, m_r) \\ U(rm+1; m_1, m_2, \cdots, m_r) = U(rm; m_1, m_2, \cdots, m_r) \\ \qquad\qquad \vdots \qquad + U(r(m-1)+1; m_1, m_2 - 1, \cdots, m_r) \\ U(rm+r-1; m_1, m_2, \cdots, m_r) = U(rm+r-2; m_1, m_2, \cdots, m_r) \\ \qquad\qquad + U(r(m-1)+r-1; m_1, m_2, \cdots, m_r - 1)\ . \end{cases}$$

Obviously, the boundary conditions given below can easily be checked. These are:

$$(7)\quad \begin{cases} \text{For } n < r \text{ or for any } m_i < 0, \\ \qquad U(n; m_1, m_2, \cdots, m_r) = 0\ ; \\ \text{for } r \le n < 2r \text{ (i.e., for } m = 1), \\ \qquad U(n; m_1, m_2, \cdots, m_r) \qquad = \begin{cases} 1 & \text{when } m_1 = m_2 = \cdots = m_r \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

Theorem 1.

$$(8)\qquad U(n;m_1,m_2,\cdots,m_r) = \begin{cases} \prod_{k=1}^{r}\binom{M+m_k}{m_k} & \text{if } n = rm + (r-1)\,, \\ \prod_{k=1}^{j+1}\binom{M+m_k}{m_k}\left[\prod_{k=j+2}^{r}\binom{M+m_k-1}{m_k}\right] & \text{if } n=rm+j \text{ and } 0 \le j \le r-2, \end{cases}$$

where

$$M = m - 1 - \sum_{i=1}^{r} m_i \,,$$

and

$$\binom{x}{y}$$

has the usual meaning with

$$\binom{x}{y} = 1 \quad\text{and}\quad \binom{x}{y} = 0$$

when $y < 0$ or when $y > x$.

Proof. Trivially, the results are true when $m = 1$. We can also verify these expressions for $m = 2$. Assume that these are valid for $m \le m'$. Now,

$$\begin{aligned} U(r(m'+1);\, m_1,m_2,\cdots,m_r) &= U(rm'+r-1;\, m_1,m_2,\cdots,m_r) + U(rm';\, m_1-1,\, m_2,\cdots,m_r) \\ &= \prod_{k=1}^{r}\binom{M'+m_k}{m_k} + \binom{M'+m_1}{m_1-1}\prod_{k=2}^{r}\binom{M'+m_k}{m_k}\,, \end{aligned}$$

$$\text{where } M' = m' - 1 - \sum_{i=1}^{r} m_i$$

$$= \binom{M'+m_1+1}{m_1}\prod_{k=2}^{r}\binom{M'+m_k}{m_k}$$

which establishes (8) for $m = m' + 1$ and $j = 0$. Similar verifications for $m' + 1$ and $0 < j \leqq r - 1$ complete the proof of Theorem 1.

Denoting $\Sigma'_{\mu, r}$ as the summation over $m_1, m_2, \cdots, m_r$ with the restriction

$$\sum_{i=1}^{r} m_i = \mu ,$$

we get the following:

Corollary 1.

$$\Sigma'_{\mu,r} U(n;m_1, m_2, \cdots, m_r) = \binom{n - (r-1)\mu - r}{\mu} . \tag{9}$$

Proof. By induction, we shall prove the result, which is seen to be true for small values of n.

$$\sum_{\mu,r} U(rm + j;\ m_1, m_2, \cdots, m_r) \tag{10}$$

$$= \sum_{\nu=0}^{\mu} \binom{m-\nu-2}{\mu-\nu} \Sigma'_{\nu, r-1} \prod_{k=1}^{j+1} \binom{m-\mu+m_k-1}{m_k} \left[\prod_{k=j+2}^{r-1} \binom{m-\mu+m_k-2}{m_k} \right]$$

$$= \sum_{\nu=0}^{\mu} \binom{m-\nu-2}{\mu-\nu} \Sigma'_{\nu, r-1} U((r-1)(m-\mu+\nu) + j; m_1, m_2, \cdots, m_{r-1})$$

$$= \sum_{\nu=0}^{\mu} \binom{m-\nu-2}{\mu-\nu} \binom{(r-1)(m-\mu) + j + \nu - r + 1}{\nu} , \text{ by induction hypothesis,}$$

$$= \sum_{\nu=0}^{\mu} \binom{rm + j - (r-1)\mu - r - \nu - 1}{\mu - \nu} \quad \text{by (1.13) of [5] ,}$$

$$= \binom{rm + j - (r-1)\mu - r}{\mu} .$$

In addition to (10), a check for $j = r - 1$ establishes (9).

Corollary 1 implies that the number of integers N, $F_n \le N < F_{n+1}$, which require $\mu + 1$ G. F. N. for minimal representation is the right-hand expression in (9), and this is in agreement with the value in [4] for $r = 2$.

Similar to

$$U(n; m_1, m_2, \cdots, m_r) ,$$

we introduce in the next definition

$$V(n; m_1, m_2, \cdots, m_r)$$

which corresponds to the maximal representation.

Definition 4. Define

$$V(n; m_1, m_2, \cdots, m_r)$$

as the number of positive integers N with the following properties:

(i) $$F_{n+r-1} - r < N \le F_{n+r} - r, \quad n \ge r;$$

(ii) In the maximal representation $(a_r, a_{r+1}, \cdots, a_n)$ of N, there are exactly m_i a'_α s among a's which are equal to zero, such that $\alpha \equiv i - 1$ (mod r), $i = 1, 2, \cdots, r$.

The definition is not vacuous, because of Lemma 2. As an illustration for $r = 3$, consider all N, $F_{10} - 3 < N \le F_{11} - 3$. The maximal representations of these integers are:

$$17 = F_3 + F_5 + F_6 + F_8, \quad 18 = F_4 + F_5 + F_6 + F_8, \quad 19 = F_3 + F_4 + F_5 + F_6 + F_8,$$
$$20 = F_4 + F_5 + F_7 + F_8, \quad 21 = F_3 + F_4 + F_5 + F_7 + F_8, \quad 22 = F_3 + F_4 + F_6 + F_7 + F_8,$$
$$23 = F_3 + F_5 + F_6 + F_7 + F_8, \quad 24 = F_4 + F_5 + F_6 + F_7 + F_8,$$
$$25 = F_3 + F_4 + F_5 + F_6 + F_7 + F_8 .$$

Thus,

$$V(8;0,0,0) = 1, \quad V(8;1,0,0) = 2, \quad V(8;0,1,0) = 2$$

$$V(8;0,0,1) = 1, \quad V(8;2,0,0) = 1, \quad V(8;1,1,0) = 1, \quad V(8;0,2,0) = 1.$$

Compare these with $U(10; m_1, m_2, m_3)$ and observe the correspondence, which is essentially the result in the theorem given below.

Theorem 2.

(10) $$V(n; m_1, m_2, \cdots, m_r) = \begin{cases} 0 & \text{when } n < r \\ U(n + r - 1; m_1, m_2, \cdots, m_r) & \text{otherwise} \end{cases}$$

Proof. It is readily checked from the last part of Lemma 2(ii) that $V(n; m_1, m_2, \cdots, m_r) = 1$ for every $n \geqq r$, when $m_1 = m_2 = \cdots = m_r = 0$. Therefore, we shall discuss the proof when m_i's are not simultaneously equal to zero.

Let $n = rm + j$, $m \geqq 0$ and $j = 1, 2, \cdots, r$. A direct verification of the theorem for $m = 0, 1$ is simple. Then, assume that it is true for $m \leq m'$. By induction, we have to show that it holds good for $m = m' + 1$.

Putting $j = 1$, the set of integers counted in

$$V(r(m' + 1) + 1; m_1, m_2, \cdots, m_r)$$

can be partitioned into two sets,

$$\{N_1\}, \; F_{r(m'+2)+1} - F_{r(m'+1)} - r < N_1 \leq F_{r(m'+2)+1} - F_r - r$$

and

$$\{N_2\}, \; F_{r(m'+2)+1} - F_{r(m'+1)+1} - r < N_2 \leq F_{r(m'+2)+1} - F_{r(m'+1)} - r,$$

each having property (ii) of Definition 4. By Lemma 3, we see that the maximal representation of every N_1 has $a_{r(m'+1)} = 1$ and $a_{r(m'+1)+1} = 1$. Therefore,

$$N_1^\star = N_1 - F_{r(m'+1)+1}, \qquad F_{r(m'+2)-1} - r < N_1^\star \leq F_{r(m'+2)} - F_r - r$$

has the maximal representation as that of N_1 without the last element

$$a_{r(m'+1)+1},$$

whereas m_i's corresponding to $N_1^\star$ have not changed from those corresponding to N_1. Due to this 1:1 correspondence, the number in $\{N_1\}$ is the same as that in $\{N_1^\star\}$ which is equal to

$$V(r(m' + 1); m_1, m_2, \cdots, m_r) \quad .$$

Using Lemma 3 again, we see that $\{N_2\}$ is in 1:1 correspondence with the set

$$\{N_2^\star\}, \quad F_{r(m'+1)} \leq N_2^\star < F_{r(m'+1)+1} \quad ,$$

such that in the minimal representation of $N_2^\star$, there are exactly m_i a_α's among non-zero a's including the last one, with $\alpha \equiv i - 1 \pmod r$, $i = 1, 2, \cdots, r$. The number in $\{N_2^\star\}$ is then equal to

$$U(r(m' + 1); m_1 - 1, m_2, \cdots, m_r) \quad .$$

Hence,

$$\begin{aligned} &V(r(m' + 1) + 1; \; m_1, m_2, \cdots, m_r) \\ &= V(r(m' + 1); \; m_1, m_2, \cdots, m_r) + U(r(m' + 1); m_1 - 1, m_2, \cdots, m_r) \\ &= U(r(m' + 2) - 1; m_1, m_2, \cdots, m_r) + U(r(m' + 1); m_1 - 1, m_2, \cdots, m_r) \end{aligned}$$

by induction hypothesis,

$$= U(r(m' + 2); \; m_1, m_2, \cdots, m_r)$$

by (6).

The cases for $1 < j \leq r$ can be treated analogously and thus the theorem is proved.

As a concluding remark we say that V's define a partition of the G.F.N., which in view of Theorem 2, is the same as given by U's. An application of partition is discussed elsewhere by the author.

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ERRATA

Please make the following changes in articles by C. W. Trigg, appearing in the December, 1967, Vol. 5, No. 5, issue of the Quarterly:

"Getting Primed for 1967" — p. 472: In the fifth line, replace "2669" with "2699."

"Curiosa in 1967" — pp. 473-476: On p. 473, place a square root sign over the 9 in "73 = ⋯ ."

On p. 474, (C), delete the "!" after the second 7.

On p. 474, (F), delete the "+" inside the parentheses.

On p. 475, (I), the last difference equals "999."

"A Digital Bracelet for 1967" — pp. 477-480: On page 478, line 7, replace the first "sum" with "sums."

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