# ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by A. P. Hillman
University of New Mexico, Albuquerque, N.M.

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

B-130 Proposed by Sidney Kravitz, Dover, New Jersey

An enterprising entrepreneur in an amusement part challenges the public to play the following game. The player is given five equal circular discs which he must drop from a height of one inch onto a larger circle in such a way that the five smaller discs completely cover the larger one. What is the maximum ratio of the diameter of the larger circle to that of the smaller ones so that the player has the possibility of winning?

B-131 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let $\{H_n\}$ be a generalized Fibonacci sequence, i.e., $H_0 = q$, $H_1 = p$, $H_{n+2} = H_{n+1} + H_n$. Extend, by the recursion formula, the definition to include negative subscripts. Show that if $|H_{-n}| = |H_n|$ for all n, then $\{H_n\}$ is a constant multiple of either the Fibonacci or the Lucas sequence.

B-132 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tenn.

Let u and v be relatively prime integers. We say that u belongs to the exponent d modulo v if d is the smallest positive integer such that $u^d \equiv 1 \pmod{v}$. For $n \geq 3$ show that the exponent to which $F_n$ belongs modulo $F_{n+1}$ is 2 if n is odd and 4 if n is even.

B-133 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let $r = F_{1000}$ and $s = F_{1001}$. Of the two numbers $r^s$ and $s^r$, which is the larger?

B-134 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Define the sequence $\{a_n\}$ by $a_1 = a_2 = 1$, $a_{2k+1} = a_{2k} + a_{2k-1}$, and $a_{2k} = a_k$ for $k \geq 1$. Show that

$$\sum_{k=1}^{n} a_k = a_{2n+1} - 1, \quad \sum_{k=1}^{n} a_{2k-1} = a_{4n+1} - a_{2n+1} .$$

B-135 Proposed by L. Carlitz, Duke University, Durham, North Carolina

Put

$$F'_n = \sum_{k=0}^{n-1} F_k 2^{n-k-1}, \quad L'_n = \sum_{k=0}^{n-1} L_k 2^{n-k-1} .$$

Show that, for all $n \geq 1$,

$$F'_n = 2^n - F_{n+2}, \quad L'_n = 3 \cdot 2^n - L_{n+2} .$$

## SOLUTIONS

### GENERALIZATION OF $F_nL_n = F_{2n-1} + F_{2n-2}$

B-112 Proposed by Gerald Edgar, Boulder, Colorado

Let $f_n$ be the generalized Fibonacci sequence $(a,b)$, i.e., $f_1 = a$, $f_2 = b$, and $f_{n+1} = f_n + f_{n-1}$. Let $g_n$ be the associated generalized Lucas sequence defined by $g_n = f_{n-1} + f_{n+1}$. Prove that $f_ng_n = bf_{2n-1} + af_{2n-2}$.

Composite of solutions by David Zeitlin, Minneapolis, Minnesota and Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $r$ and $s$ be the roots of $x^2 - x - 1 = 0$. Then $f_n$ and $g_n$ are of the form $c_1r^n + c_2s^n$ and hence $f_ng_n$, $f_{2n-1}$, and $f_{2n-2}$ are all of the form

$$k_1r^{2n} + k_2(-1)^n + k_3s^{2n}$$

and hence all satisfy the difference equation

(E) $$y_{n+3} - 2y_{n+2} - 2y_{n+1} + y_n = 0$$

whose auxiliary polynomial is

$$(x - r^2)(x - rs)(x - s^2) = x^3 - 2x^2 - 2x + 1.$$

Since both sides of the desired formula

(F) $$f_n g_n = bf_{2n-1} + af_{2n-2}$$

satisfy (E), formula (F) is established by verifying it for $n = 0, 1,$ and 2 and then using (E) and mathematical induction to prove it for $n \geq 0$.

Also solved by Thomas P. Dence, Douglas Lind, D. V. Jaiswal (India), Stanley Rabinowitz, A. C. Shannon (Australia), M. N. S. Swamy (Canada), and the proposer.

## CLUSTER POINTS

B-113 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let $(x)$ denote the fractional part of $x$, so that if $[x]$ is the greatest integer in $x$, $(x) = x - [x]$. Let $a = (1 + \sqrt{5})/2$ and let $A$ be the set $\{(a), (a^2), (a^3), \cdots\}$. Find all the cluster points of $A$.

Solution by the proposer.

If $b = (1 - \sqrt{5})/2$, it is familiar that $L_n = a^n + b^n$, where $L_n$ is the $n^{th}$ Lucas number, which is an integer. Since $-1 < b < 0$, given $\in > 0$, there is an $N$ such that for all $k > N$ we have

$$0 < b^{2k} = L_{2k} - a^{2k} < \in .$$

It follows that $(a^{2k}) \to 1$. Similarly, there is an $M$ such that for all $k > M$ we have

$$0 < -b^{2k+1} = a^{2k+1} - L_{2k+1} < \in ,$$

so $(a^{2k+1}) \to 0$. Clearly these are the only possible cluster points of $A$.

## OUR MAN OF PISA

B-114 Proposed by Gloria C. Padilla, University of New Mexico, Albuquerque, N.M.

Solve the division alphametic

$$\begin{array}{r} \text{PISA} \\ \text{FIB}\,\overline{)\text{ONACCI}} \end{array},$$

where each letter is one of the digits 1, 2, ···, 9 and two letters may represent the same digit. (This is suggested by Maxey Brooke's B-80.)

Solution by the proposer.

One solution is the following:

$$\begin{array}{r} 3\,4\,1\,8 \\ 1\,4\,3\,\overline{)4\,8\,8\,7\,7\,4} \end{array}$$

## IDENTITIES FOR $F_{kn}$ AND $L_{kn}$

B-115 Proposed by H. H. Ferns, Victoria, B.C., Canada

From the formulas of B-106:

$$2F_{i+j} = F_iL_j + F_jL_i$$
$$2L_{i+j} = 5F_iF_j + L_iL_j$$

one has

$$F_{2n} = F_nL_n$$
$$F_{3n} = (5F_n^3 + 3F_nL_n^2)/4$$
$$L_{2n} = (5F_n^2 + L_n^2)/2$$
$$L_{3n} = (15F_n^2L_n + L_n^3)/4$$

Find and prove the general formulas of these types.

Solution by Stanley Rabinowitz, Far Rockaway, New York.

The formulas look neater when expressed in matrix form. Putting $i = (k-1)n$ and $j = n$ in the formulas of B-106 gives

$$\text{(R)}\qquad \begin{pmatrix} F_{kn} \\ L_{kn} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} L_n & F_n \\ 5F_n & L_n \end{pmatrix}\begin{pmatrix} F_{(k-1)n} \\ L_{(k-1)n} \end{pmatrix}$$

Repeated application of this formula gives the desired solution:

$$\begin{pmatrix} F_{kn} \\ L_{kn} \end{pmatrix} = \frac{1}{2^k}\begin{pmatrix} L_n & F_n \\ 5F_n & L_n \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

since $F_0 = 0$ and $L_0 = 2$.

Note: From (R) or the formulas of B-106, one can obtain the proposer's formulas:

$$F_{(k+1)n} = \frac{1}{2^k}\sum_{i=0}^{[k/2]} 5^i \binom{k+1}{k-2i} F_n^{2i+1} L_n^{k-2i} ,$$

$$L_{(k+1)n} = \frac{1}{2^k}\sum_{i=0}^{[(k+1)/2]} 5^i \binom{k+1}{k+1-2i} F_n^{2i} L_n^{k+1-2i} .$$

Also solved by David Zeitlin and the proposer.

## A GENERATING FUNCTION

B-116 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Find a compact sum for the series

$$\sum_{m,n=0}^{\infty} F_{2m-2n} x^m y^n .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

If $W_{n+2} = aW_{n+1} + bW_n$, then

$$\frac{W_0 + (W_1 - aW_0)t}{1 - at - bt^2} = \sum_{k=0}^{\infty} W_k t^k \quad . \tag{1}$$

Since $W_k = F_{2k \pm p}$ satisfies $W_{k+2} = 3W_{k+1} - W_k$, we have

$$\sum_{m=0}^{\infty} F_{2m-2n} x^m = \frac{F_{-2n} + (F_{2-2n} - 3F_{-2n})x}{1 - 3x + x^2} \quad .$$

Since $F_{-j} = (-1)^{j+1} F_j$, we have the desired sum, S,

$$S = \frac{1}{1 - 3x + x^2}\left((3x - 1)\sum_{n=0}^{\infty} F_{2n} y^n - x\sum_{n=0}^{\infty} F_{2n-2} y^n\right)$$

$$= \frac{1}{1 - 3x + x^2}\left(\frac{(3x - 1)y}{1 - 3y + y^2} - \frac{x(-1 + 3y)}{1 - 3y + y^2}\right)$$

$$= \frac{x - y}{(1 - 3x + x^2)(1 - 3y + y^2)}$$

Also solved by Douglas Lind, D. V. Jaiswal (India), M.N.S. Swamy (Canada), and the proposer.

## ANOTHER GENERATING FUNCTION

B-117 Proposed by L. Carlitz, Duke University, Durham, No. Carolina.

Find a compact sum for the series

$$\sum_{m,n=0}^{\infty} F_{2m-2n+1} x^m y^n \quad .$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Using (1) in B-116, we have

$$\sum_{m=0}^{\infty} F_{2m-2n+1}x^m = \frac{F_{-2n+1} + (F_{3-2n} - 3F_{-2n+1})x}{1 - 3x + x^2}$$

Since $F_{-j} = (-1)^{j+1} F_j$, we have the desired sum, S,

$$S = \frac{1}{1 - 3x + x^2}\left((1 - 3x)\sum_{n=0}^{\infty} F_{2n-1}y^n + x\sum_{n=0}^{\infty} F_{2n-3}y^n\right)$$

$$= \frac{1}{1 - 3x + x^2}\left(\frac{(1 - 3x)(1 - 2y)}{1 - 3y + y^2} + \frac{x(2 - 5y)}{1 - 3y + y^2}\right)$$

$$= \frac{xy - 2y - x + 1}{(1 - 3x + x^2)(1 - 3y + y^2)} .$$

Also solved by Douglas Lind, D. V. Jaiswal (India), M.N.S. Swamy (Canada), and the proposer.

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