A MAGIC SQUARE INVOLVING FIBONACCI NUMBERS

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Is there anything peculiar in the magic square

13	89	97	34
110	21	63	39
68	94	55	16
42	29	18	144 ?

As is well known, the arrangement derives its name from the property that the sum of all numbers contained in either a row, or a column, or a diagonal is a constant, in this case 233, which we will refer to as the "magic number," MN.

It seems that knowledge of such square number arrays, not necessarily four-by-four, were known in China as far back as 2200 B.C., but apparently it was only in the 16th Century that this idea has reached the Christian Occident. From that time onward, this topic has not only held its attraction in a recreational manner, but mathematicians of rank, among them Leonhard Euler, have given it serious attention.

Here is a suggestion for designing a whole set of four-by-four magic squares with the added property that each of the entries is the sum of two Fibonacci numbers (which may, of course, under appropriate conditions, be a Fibonacci number itself) and the magic number is itself a Fibonacci number — even a pre-assigned one, if one so desires.

We will first construct an addition table, denoting by f_{n_i}, $i \in \{1, 2, \cdots, 8\}$, the n_i^{th} Fibonacci number.

If we wish to retain the positions of the numbers in the main diagonal

$$MN = \sum_{i=1}^{8} f_{n_i}$$

	f_{n_5}	f_{n_6}	f_{n_7}	f_{n_8}
f_{n_1}	$f_{n_1} + f_{n_5}$	$f_{n_1} + f_{n_6}$	$f_{n_1} + f_{n_7}$	$f_{n_1} + f_{n_8}$
f_{n_2}	$f_{n_2} + f_{n_5}$	$f_{n_2} + f_{n_6}$	$f_{n_2} + f_{n_7}$	$f_{n_2} + f_{n_8}$
f_{n_3}	$f_{n_3} + f_{n_5}$	$f_{n_3} + f_{n_6}$	$f_{n_3} + f_{n_7}$	$f_{n_3} + f_{n_8}$
f_{n_4}	$f_{n_4} + f_{n_5}$	$f_{n_4} + f_{n_6}$	$f_{n_4} + f_{n_7}$	$f_{n_4} + f_{n_8}$

Array 1

Therefore, we must rearrange the positions of the other numbers. The minor diagonal already does show the desired property. Studying the design of our array, a grouping of the non-diagonal numbers into pairs of positions, which are symmetric with respect to the center of the square, as indicated by the guide lines, suggests itself. We now perform an interchange of the two numbers in each pair. After this transformation, our array becomes:

$f_{n_1} + f_{n_5}$	$f_{n_4} + f_{n_7}$	$f_{n_4} + f_{n_6}$	$f_{n_1} + f_{n_8}$
$f_{n_3} + f_{n_8}$	$f_{n_2} + f_{n_6}$	$f_{n_2} + f_{n_7}$	$f_{n_3} + f_{n_5}$
$f_{n_2} + f_{n_8}$	$f_{n_3} + f_{n_6}$	$f_{n_3} + f_{n_7}$	$f_{n_2} + f_{n_5}$
$f_{n_4} + f_{n_5}$	$f_{n_1} + f_{n_7}$	$f_{n_1} + f_{n_6}$	$f_{n_4} + f_{n_8}$

Array 2

Then MN, as computed from the first row, equals

$$2(f_{n_1} + f_{n_4}) + f_{n_5} + f_{n_6} + f_{n_7} + f_{n_8} .$$

For this number, however, to equal

$$\sum_{i=1}^{8} f_{n_i} ,$$

the condition

$$f_{n_1} + f_{n_4} = f_{n_2} + f_{n_3} ,$$

forces itself upon us. Analogously, we would need to stipulate that

$$f_{n_5} + f_{n_8} = f_{n_6} + f_{n_7} .$$

If we wish to obtain an array which has the properties of a magic square without further restrictions, an interchange of the subset of four numbers which form a square arrangement at the right-hand top of Array 2 with the corresponding one on the left-hand bottom needs to be resorted to. This transformation leads to:

$f_{n_1} + f_{n_5}$	$f_{n_4} + f_{n_7}$	$f_{n_2} + f_{n_8}$	$f_{n_3} + f_{n_6}$
$f_{n_3} + f_{n_8}$	$f_{n_2} + f_{n_6}$	$f_{n_4} + f_{n_5}$	$f_{n_1} + f_{n_7}$
$f_{n_4} + f_{n_6}$	$f_{n_1} + f_{n_8}$	$f_{n_3} + f_{n_7}$	$f_{n_2} + f_{n_5}$
$f_{n_2} + f_{n_7}$	$f_{n_3} + f_{n_5}$	$f_{n_1} + f_{n_6}$	$f_{n_4} + f_{n_8}$

Array 3

Array 3 gives us a prescription for an infinite set of magic squares, such that each entry is the sum of (at most) two Fibonacci numbers.

However, we may wish to have a Fibonacci number for our MN. We need to superimpose the set of conditions: $n_i = n_1 + 2i - 3, \quad i \in \{2, 3, \cdots, 8\}$. Now our magic square reads

$f_a + f_{a+7}$	$f_{a+5} + f_{a+11}$	$f_{a+1} + f_{a+13}$	$f_{a+3} + f_{a+9}$
$f_{a+3} + f_{a+13}$	$f_{a+1} + f_{a+9}$	$f_{a+5} + f_{a+7}$	$f_a + f_{a+11}$
$f_{a+5} + f_{a+9}$	$f_a + f_{a+13}$	$f_{a+3} + f_{a+11}$	$f_{a+1} + f_{a+7}$
$f_{a+1} + f_{a+11}$	$f_{a+3} + f_{a+7}$	$f_a + f_{a+9}$	$f_{a+5} + f_{a+13}$

Array 4

where, for simplicity's sake, we symbolized n_1 by a. Array 4 gives us the means for structuring an infinitude of magic squares. Obviously, we may even pre-assign a magic number, we wish to obtain. Since MN now becomes f_{a+14}, the n^{th} Fibonacci number will be obtained by letting the parameter a equal n - 14.

But we may superimpose one more restriction. Our aim now is to contrive a magic square of the kind just described but with the added property that all elements in the major diagonal are Fibonacci numbers, rather than sums of two Fibonacci numbers. Then — see Array 3 — the restrictions:

$$n_2 = n_5 = n_1 + 1$$
$$n_3 = n_1 + 3$$
$$n_4 = n_1 + 5$$
$$n_6 = n_1 + 2$$
$$n_7 = n_1 + 4$$
$$n_8 = n_1 + 5$$

need to be observed, and the arrangement below (Array 5) will serve our needs. Again, we adopt the simplified symbolism.

f_{a+2}	f_{a+6}	$f_{a+1} + f_{a+6}$	f_{a+4}
$f_{a+3} + f_{a+6}$	f_{a+3}	$f_{a+1} + f_{a+5}$	$f_a + f_{a+4}$
$f_{a+2} + f_{a+5}$	$f_a + f_{a+6}$	f_{a+5}	$2f_{a+1}$
$f_{a+1} + f_{a+4}$	$f_{a+1} + f_{a+3}$	$f_a + f_{a+2}$	f_{a+7}

Array 5

The magic number $MN = f_{a+8}$ is again pre-assignable, and all entries in our major diagonal are Fibonacci numbers. We may test our scheme by wishing to obtain the 13^{th} Fibonacci number, 233, as our magic number. Then $a = 5$, and our magic square becomes the one quoted at the beginning of this paper.

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