FACTORIZATION OF 2 X 2 INTEGRAL MATRICES WITH DETERMINANT ±1

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1. CANONICAL PRODUCTS AND CANONICAL REPRESENTATIVES

Let Z denote the integers and $M_2(Z)$

$$M_2(Z) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in Z \right\},$$

the set of 2×2 integral matrices. The matrices of $M_2(Z)$ which have inverses in $M_2(Z)$ are denoted by $GL(2, Z)$, i.e.,

$$GL(2, Z) = \left\{ x \in M_2(Z) : \det x = \pm 1 \right\} .$$

We shall develop an algorithm which uses various properties of the Fibonacci numbers for expressing any element of $GL(2, Z)$ as a product of powers of the matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} .$$

This of course implies that A and B generate $GL(2, Z)$, a result which has been noted elsewhere [3]. The algorithm forms part of the author's B. A. thesis written under the direction of B. Hunt at Reed College in 1957.

1.1 Definition: A "canonical product" is any product of the form

$$U = A^{a_n} B^{b_n} A^{a_{n-1}} B^{b_{n-1}} \cdots A^{a_2} B^{b_2} A^{a_1} B^{b_1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

where $n \geq 1$ and $a_i \geq 0$, $b_i \geq 0$ where we assume strict inequality except possibly at $i = n$ and $i = 1$ respectively.

We note that

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix},$$

where F_n is the n^{th} Fibonacci number of the following sequence:

n:	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
F_n:	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584

1.2 Theorem. If

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is any canonical product then $a \geq c \geq 0$, $b \geq d \geq 0$.

Proof. The theorem is true for A and B the products with one factor. Suppose the theorem true for any product

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of k factors. Then

$$AT = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} \text{ and } \begin{matrix} a+c \geq c \geq 0 \\ b+d \geq d \geq 0. \end{matrix}$$

$$BT = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ a & b \end{pmatrix} \text{ and } \begin{matrix} a+c \geq a \geq 0 \\ b+d \geq b \geq 0. \end{matrix}$$

hence the throrem holds for any product of $k+1$ factors and hence, by induction, for any U.

1.3 Corollary: Not both c and d are zero and

i) $a > c \geq 0$ unless $U = \begin{pmatrix} 1 & n+1 \\ 1 & n \end{pmatrix} = BA^n$, $n \geq 0$

ii) $b > d \geq 0$ unless $U = \begin{pmatrix} n+1 & 1 \\ n & 1 \end{pmatrix} = BA^{n-1}B$, $n \geq 0$

iii) $c \neq d$ unless $U = \begin{pmatrix} n+1 & n \\ 1 & 1 \end{pmatrix} = A^{n-1}B^2, \quad n \geq 1$

or

$U = \begin{pmatrix} n & n+1 \\ 1 & 1 \end{pmatrix} = A^{n-1}BA, \quad n \geq 1$

iv) $a \neq b$ unless $U = A$ or B.

v) $c > 0$ unless $U = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = A^n$

vi) $d < 0$ unless $U = \begin{pmatrix} n & 1 \\ 1 & 0 \end{pmatrix} = A^{n-1}B$.

Proof. These are immediate consequences of the theorem and the fact that $\det U = ad - bc = \pm 1$.

1.4 Corollary: If

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then

i) $a - b > 0$ implies $c - d \geq 0$

ii) $a - b < 0$ implies $c - d \leq 0$.

Proof. i) $a > b$, $c \leq d$ implies

$$ad - bc > bd - bc > bc - bc = 0$$

Hence $ad - bc \geq 2$, which is impossible.

ii) $a < b$, $c > d$ implies $b > a \geq c > d \geq 0$ and so $b \geq 2$. Hence $ad - bc < bd - bc = b(d - c) \leq 2(-1) = -2$ which is impossible.

1.5 Theorem. Let

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be a canonical product. Then

1. $a - b = 0$ implies $U = A, B$.
2. $a - b < 0$ implies U ends in A and $a - b \le c - d \le 0$
3. $a - b > 0$ implies U ends in B and $a - b \ge c - d \ge 0$.

<u>Proof.</u> 1 follows from Corollary 1.3.

2 and 3. The theorem is immediately verified for products of two factors

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad AB = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } BA = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

Suppose the theorem holds for products with k or fewer factors and

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a product with k factors where $k \ge 2$. Then

$$UB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a+b & a \\ c+d & c \end{pmatrix}$$

and

$$(a + b) - a = b \ge 1 > 0 \quad \text{and} \quad b \ge (c + d) - c = d \ge 0 .$$

Note $b = 0$ implies $d = 0$ which contradicts $\det U = \pm 1$. Likewise,

$$UA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

hence

$$a - (a + b) = -b < 0 \text{ and } c - (c + d) = -d \text{ and } -b \le -d \le 0.$$

Thus the theorem holds for products of $k + 1$ factors and hence for all canonical products by induction.

Theorem 1.2 says that if

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a canonical product then $a \geq c \geq 0$, and $b \geq d \geq 0$ while Theorem 1.5 allows one to decide if the canonical product ends in an A or B. Not every unimodular matrix satisfies the conditions of Theorem 1.2 but the following theorem characterizes the situation.

<u>1.6 Theorem.</u> (See [2].) Any matrix

$$R = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in GL(2, Z)$$

different from I, A, B can, by suitable multiplications by powers of A and B, be brought to the form

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ,$$

where (a, b, c, d) is some permutation of $(|r|, |s|, |t|, |u|)$ and $a \geq c \geq 0$, $b \geq d \geq 0$. U is called a canonical representative of R.

<u>Proof.</u> From the condition $ru - st = \pm 1$ we can conclude that no three of the quantities r, s, t, u can be negative and the remaining one positive or three of them positive and one negative. There are therefore three remaining cases:

1. r, s, t, u are all non-negative,
2. r, s, t, u are all non-positive,
3. Two of r, s, t, u are negative, two are non-negative.

In case 2 we note

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} -r & -s \\ -t & -u \end{pmatrix}.$$

In case 3 we note

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} -r & -s \\ t & u \end{pmatrix}$$

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -r & s \\ -t & u \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -r & s \\ t & -u \end{pmatrix}$$

While

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} s & r \\ u & t \end{pmatrix}.$$

The multipliers can be expressed as follows:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A^{-1}B\,, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = A^{-1}B^{-1}ABA^{-1}\,,$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus we have that

$$M\begin{pmatrix} r & s \\ t & u \end{pmatrix}N = \begin{pmatrix} |r| & |s| \\ |t| & |u| \end{pmatrix},$$

where M and N are suitable products of powers of A and B. Note $|r|\,|u| - |s|\,|t| = |ru| - |st| = \pm 1$ since $1 = |ru - st| \geq ||ru| - |st||$ and $|ru| = \pm st$ is not possible. Also operating with

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we can bring any element, in particular the largest, to the upper left position. If

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq A, B$$

is in this form, i.e., $a \geq b$, $cd \geq 0$ then we may assume that $b \geq d$. For $b < d$ implies $ad - bc \geq ad - cd = (a - c)d \geq 0$ unless $a = c$. If $a = c$ then

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \text{ has the property } \begin{matrix} c \geq a \geq 0 \\ d \geq b \geq 0 \end{matrix} \quad .$$

Every unimodular matrix has 2 canonical representatives depending on whether a maximal element is brought to the upper left or upper right-hand corner. We now prove the converse of Theorem 1.2.

<u>1.7 Theorem.</u> Every canonical representative of a unimodular 2×2 matrix is a canonical product.

<u>Proof.</u> Let

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be a canonical representative, i.e., $a \geq c \geq 0$, $b \geq d \geq 0$. If the largest element of $U = 1$ then $U = A, B$ and the theorem holds. Assume the theorem is true for $\max(a, b) \leq r$, where $r \geq 1$. Then there are two possibilities for $\max(a, b) = r + 1$:

<u>Case 1.</u>

$$Y = \begin{pmatrix} r+1 & b \\ c & d \end{pmatrix};$$

<u>Case 2.</u>

$$Z = \begin{pmatrix} a & r+1 \\ c & w \end{pmatrix} \quad .$$

We now analyze Case 1. We have that $0 \le c, b, d < r + 1$ otherwise $r + 1 \ge 2$ divides $\det Y = \pm 1$.

$$YB^{-1} = \begin{pmatrix} r+1 & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} b & r+1-b \\ d & c-d \end{pmatrix}.$$

If $d = 0$ then $b = c = 1$ and

$$YB^{-1} = \begin{pmatrix} b & r \\ d & c \end{pmatrix} = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} = A^r$$

Hence $Y = A^rB$ is a canonical product. If $d \ne 0$ then $c - d \ge 0$ for otherwise

$$\left|\det (YB^{-1})\right| \ge 2 .$$

We need only establish that $r + 1 - b \ge c - d$ to show that YB^{-1} is a canonical representative. If

$$\det Y = (r + 1)d - bc = 1$$

then

$$(r + 1)d - bd = 1 + bc - bd$$

and so

$$(r + 1) - b = \frac{1}{d} + \frac{b}{d}(c - d) \ge c - d$$

since $b \ge d$. If

$$\det Y = (r + 1)d - bc = -1$$

then $b, c > d$ since $b, c < r + 1$.

Hence

$$(r + 1) - b = \frac{b}{d}(c - d) - \frac{1}{d} = (c - d) + \left(\frac{b}{d} - 1\right)(c - d) - \frac{1}{d}$$

$$= (c - d) + \frac{1}{d}\left[(b - d)(c - d) - 1\right] \geq c - d.$$

Thus YB^{-1} is a canonical product by induction hypothesis. This implies that Y must also be a canonical product.

Case 2 is analyzed by an analogous treatment of ZA^{-1}. The theorem then follows by induction.

2. THE ALGORITHM

Let $\overline{U} \in GL(2, Z)$. Theorem 1.6 describes how to obtain a canonical representative U for $\overline{U}$. Theorem 1.7 asserts that U is a canonical product and Theorem 1.5 establishes whether U ends in an A or B. The following theorems provide a quantitative counterpart for Theorem 1.5.

2.1 Theorem. If

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is a canonical product and $a - b < 0$ then $U = U_1A^n$ where

$$N = \left[\frac{b}{a}\right]$$

and U_1 ends in B.*

Proof. If $a = 1$ we consult Corollary 1.3. If $a \neq 1$ we note:

1. $b \neq na$ since $b = na$ implies $a | \det U = \pm 1$ and $a = 1$,
2. $a > c$ since $a = c$ implies $a = 1$. Then

*$[\alpha]$ is the greatest integer less than or equal to α

$$UA^{-n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -na+b \\ c & -nc+d \end{pmatrix} = U_1 \,.$$

Since

$$N = \left[\frac{b}{a}\right]$$

$$a - (-na + b) > 0 \,.$$

Hence if U_1 is a canonical product it ends in a B. If $-nc + d < 0$ then, recalling that $a \geq 2$,

$$\det U_1 = a(-nc + d) - c(-na + b) < -2 - c$$

which is impossible. Hence $-nc + d \geq 0$. Further

$$r = -nc + d \leq -na + b$$

since

$$r > -na + b \geq 0$$

implies

$$\begin{aligned} \det U_1 &= ar - c(-na + b) \geq ar = (a - 1)(r - 1) \\ &= ar - ar + (a + r) - 1 \geq 2 + 1 - 1 = 2 \end{aligned}$$

which is impossible.

2.2 Lemma.

1. $$\frac{F_n}{F_{n-1}} < \frac{a}{b} < \frac{F_{n+2}}{F_{n+1}} < \frac{\sqrt{5}+1}{2} \quad \text{implies } b > F_{n+1}$$

   if n is even,

2. $$\frac{\sqrt{5}+1}{2} < \frac{F_{n+2}}{F_{n+1}} < \frac{a}{b} < \frac{F_n}{F_{n-1}} \quad \text{implies } b > F_{n+1}$$

if n is odd.

Proof. We prove 1 using continued fraction notation [1, Chapter X]. In the following the initial block of ones in the continued fraction symbols will be of length $n - 1$. Also we note

$$[x_0, x_1, \cdots, x_n, 1] = [x_0, x_1, \cdots, x_n + 1] \quad .$$

$$\frac{F_n}{F_{n-1}} = [1, 1, \cdots, 1]$$

is a convergent to

$$\frac{\sqrt{5} + 1}{2} = [1, 1, \cdots] \quad .$$

If we express

$$\frac{a}{b} = [a_0, a_1, \cdots, a_m]$$

as a continued fraction then by the continued fraction algorithm [1, p. 140]

$$\frac{a}{b} = [1, 1, \cdots, 1, a_{n-1}, a_n, a_m] \text{ since } \frac{F_n}{F_{n-1}} < \frac{a}{b} < \frac{F_{n+2}}{F_{n+1}} \, ,$$

i.e., $m \geq n - 1$ and $a_0 = a_1 = \cdots = a_{n-2} = 1$.

Letting

$$[a_{n-1}, a_n, \cdots, a_m] = a'_{n-1} = \frac{r}{s} \quad ,$$

where $(r, s) = 1$ we have

$$\frac{F_n}{F_{n-1}} = [1, 1, \cdots, 1] < \left[1, 1, \cdots, 1, \frac{r}{s}\right] < [1,1,\cdots,1,2] = \frac{F_{n+2}}{F_{n+1}} .$$

Hence

$$\frac{a}{b} = \frac{rF_n + sF_{n-1}}{rF_{n-1} + sF_{n-2}} = \frac{\frac{r}{s}F_n + F_{n-1}}{\frac{r}{s}F_{n-1} + F_{n-2}} < \frac{2F_n + F_{n-1}}{2F_{n-1} + F_{n-2}} = \frac{F_{n+2}}{F_{n+1}} .$$

By the continued fraction algorithm

$$a'_{n-1} = \frac{r}{s} \geq 1 .$$

Now $r/s = 1$ implies

$$\frac{a}{b} = \frac{F_{n+1}}{F_n} > \frac{\sqrt{5}+1}{2} .$$

Hence $r/s > 1$ and $r \geq 2$. Likewise, $r = 2$ implies $s = 1$ and

$$\frac{a}{b} = \frac{F_{n+2}}{F_{n+1}} .$$

Hence $r \geq 2$. If

$$(rF_n + sF_{n-1}, rF_{n-1} + sF_{n-2}) = 1$$

then

$$b \geq rF_{n-1} + sF_{n-2} > F_{n+1} = 2F_{n-1} + F_{n-2} .$$

But

$$d \mid (rF_n + sF_{n-1}), \quad d \mid (rF_{n-1} + sF_{n-2})$$

implies

$$d \mid \{r(F_n - F_{n-1}) + s(F_{n-1} - F_{n-2})\} = rF_{n-2} + sF_{n-3}$$

Hence

$$d \mid (rF_1 + F_0) = r \quad ,$$

Also

$$d \mid r, \quad d \; (rF_k + sF_{k-1}), \quad 1 \le k \le n$$

implies $d \mid sF_{k-1}$ and hence

$$d \mid F_{k-1}, \quad 1 \le k \le n \quad ,$$

since $(r, s) = 1$. Thus $d = 1$, since the F_k are relatively prime, a fact which can be established in the same recursive manner.

The same type of argument is used in proving 2.

2.3 Theorem. If

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a - b > 0, \quad b > d > 0$$

then $U = U_1B^n$, where n is determined by locating a/b with respect to the sequence of points

$$\frac{F_2}{F_1} < \frac{F_4}{F_3} < \frac{F_6}{F_5} < \cdots < \frac{\sqrt{5}+1}{2} < \cdots < \frac{F_7}{F_6} < \frac{F_5}{F_4} < \frac{F_3}{F_2}$$

and U_1 is a canonical product ending in A or $U_1 = A, B$. More precisely,

(1) $$\frac{F_n}{F_{n-1}} < \frac{a}{b} \le \frac{F_{n+2}}{F_{n+1}} \quad \text{if } n \text{ is even ;}$$

(2) $$\frac{F_{n+2}}{F_{n+1}} \le \frac{a}{b} < \frac{F_n}{F_{n-1}}, \text{ if } n \text{ is odd};$$

(3) $$2 = \frac{F_3}{F_2} \le \frac{a}{b}, \text{when } n = 1.$$

Finally, if $b = d = 1$ or $d = 0$ we consult Corollary 1.3.

Proof. We prove 1 and note that the proofs to 2 and 3 are analogous. Suppose

$$\frac{F_n}{F_{n-1}} < \frac{a}{b} \le \frac{F_{n+2}}{F_{n+1}} < \frac{\sqrt{5}+1}{2}$$

Then

$$U_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} B^{-n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (-1)^n \begin{pmatrix} F_{n-1} & -F_n \\ -F_n & F_{n+1} \end{pmatrix}$$

$$= \begin{pmatrix} a\ F_{n-1} - b\ F_n & -aF_n + b\ F_{n+1} \\ c\ F_{n-1} - d\ F_n & -cF_n + d\ F_{n+1} \end{pmatrix}.$$

We first note that

$$(aF_{n-1} - bF_n) - (-aF_n + bF_{n+1}) = aF_{n+1} - bF_{n+2} \le 0.$$

Hence, if we establish that U_1 is a canonical product, then U_1 ends in A or $U_1 = A_2B$. To show tnat U_1 is a canonical product we note that $aF_{n-1} - bF_n \ge 0$ and $-aF_n + bF_{n+1} > 0$, since

$$\frac{F_n}{F_{n-1}} < \frac{a}{b} \le \frac{F_{n+2}}{F_{n+1}} < \frac{F_{n+1}}{F_n}.$$

This shows that the top row has positive entries. If we show

(i) $$\frac{F_n}{F_{n-1}} \le \frac{c}{d} \le \frac{F_{n+1}}{F_n}$$

(ii) $$\frac{F_n}{F_{n-1}} \le \frac{a-c}{b-d} \le \frac{F_{n+1}}{F_n}$$

then the bottom row is positive and the columns decrease since the two equations

$$(aF_{n-1} - bF_n) - (cF_{n-1} - dF_n) = (a-c)F_{n-1} - (b-d)F_n \ge 0$$

$$(-aF_n + bF_{n+1}) - (-cF_n + dF_{n+1}) = -(a-c)F_n + (b-d)F_{n+1} \ge 0$$

are equivalent to (ii). We now note that

$$\left|\frac{a}{b} - \frac{c}{d}\right| = \left|\frac{ab - bc}{bd}\right| = \frac{1}{bd}$$

for use in proving (i),

$$\left|\frac{a}{b} - \frac{a-c}{b-d}\right| = \left|\frac{ab - ad - ab + bc}{b(b-d)}\right| = \frac{1}{b(b-d)}$$

for proving (ii). We conclude by proving (i) since (ii) is similar.

Since

$$\frac{F_n}{F_{n-1}} < \frac{a}{b} \le \frac{F_{n+2}}{F_{n+1}}, \quad b \ge F_{n+1}$$

by Lemma 2.2. If

$$\frac{c}{d} < \frac{F_n}{F_{n-1}}$$

then

$$\frac{1}{dF_{n-1}} > \frac{1}{dF_{n-1}} \geq \frac{1}{bd} = \left|\frac{a}{b} - \frac{c}{d}\right| > \left|\frac{F_n}{F_{n-1}} - \frac{c}{d}\right| > \frac{1}{dF_{n-1}}$$

which is impossible. Hence

$$\frac{F_n}{F_{n-1}} \leq \frac{c}{d} \; .$$

Likewise

$$\frac{c}{d} > \frac{F_{n+1}}{F_n}$$

implies

$$\frac{1}{dF_n} > \frac{1}{dF_{n+1}} \geq \frac{1}{bd} = \frac{c}{d} - \frac{a}{b} \geq \frac{c}{d} - \frac{F_{n+1}}{F_n} \geq \frac{F_{n+1}}{F_n} \geq \frac{F_{n+2}}{F_{n+1}}$$

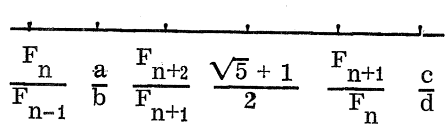
$$\geq \frac{1}{dF_n} + \frac{1}{F_nF_{n+1}}$$

which is impossible since

$$\frac{1}{F_nF_{n+1}} > 0 \; .$$

Hence

$$\frac{c}{d} \leq \frac{F_{n+1}}{F_n} \; .$$



3. EXAMPLE

$$U = \begin{pmatrix} 206 & 1575 \\ 79 & 604 \end{pmatrix}$$

is a canonical product ending in A^7, since $a - b = 206 - 1575 < 0$ and

$$\left[\frac{1575}{206}\right] = 7 \; .$$

So that

$$\begin{pmatrix} 206 & 1575 \\ 79 & 604 \end{pmatrix}\begin{pmatrix} 1 & -7 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 206 & 133 \\ 79 & 51 \end{pmatrix} = U_1$$

Then,

$$\frac{F_4}{F_3} = 1.5 < \frac{a}{b} = \frac{206}{133} = 1.55 \leq 1.6 = \frac{F_6}{F_5}$$

hence U_1 ends in B^4. We note that

$$\begin{pmatrix} 206 & 133 \\ 79 & 51 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} = \begin{pmatrix} 13 & 47 \\ 5 & 18 \end{pmatrix}$$

ends in A^3. Since

$$\left[\frac{47}{13}\right] = 3 \; .$$

$$\begin{pmatrix} 13 & 47 \\ 5 & 18 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 8 \\ 5 & 3 \end{pmatrix} = U_3$$

$$\frac{F_7}{F_6} = \frac{a}{b} = \frac{13}{8} = 1.625 \leq 1.6667 = \frac{F_5}{F_4}$$

hence U_3 ends in B^5.

$$\begin{pmatrix} 13 & 8 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} -3 & 5 \\ 5 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = A .$$

Therefore

$$U = AB^5A^3B^4A^7 .$$

4. TABLE

$\frac{F_3}{F_2} = \frac{2}{1} = 2.0000000$	$\frac{F_{18}}{F_{17}} = \frac{2584}{1597} = 1.6180338$
$\frac{F_5}{F_4} = \frac{5}{3} = 1.6666667$	$\frac{F_{16}}{F_{15}} = \frac{987}{610} = 1.6180328$
$\frac{F_7}{F_6} = \frac{13}{8} = 1.6250000$	$\frac{F_{14}}{F_{13}} = \frac{377}{233} = 1.6180258$
$\frac{F_9}{F_8} = \frac{34}{21} = 1.6190476$	$\frac{F_{12}}{F_{11}} = \frac{144}{89} = 1.6179775$
$\frac{F_{11}}{F_{10}} = \frac{89}{55} = 1.6181818$	$\frac{F_{10}}{F_9} = \frac{55}{34} = 1.6176471$
$\frac{F_{13}}{F_{12}} = \frac{233}{144} = 1.6180556$	$\frac{F_8}{F_7} = \frac{21}{13} = 1.6153846$
$\frac{F_{15}}{F_{14}} = \frac{610}{377} = 1.6180371$	$\frac{F_6}{F_5} = \frac{8}{5} = 1.6000000$
$\frac{F_{17}}{F_{16}} \quad \frac{1597}{987} = 1.6180344$	$\frac{F_4}{F_3} = \frac{3}{2} = 1.5000000$
$\frac{\sqrt{5} + 1}{2} = 1.6180340$	$\frac{F_2}{F_1} = \frac{1}{1} = 1.0000000$

REFERENCES

1. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Clarendon, Oxford, 1938.

2. Hunt Burrowes, "Continued Fractions and Indefinite Forms," unpublished article, Notice in Bull. Am. Math. Soc. 62 (1956), 553.

3. George K. White, "On Generators and Defining Relations for the Unimodular Group M_2," Amer. Math. Month. 71 (1964), 743-748.

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ERRATA

Please make the following corrections in "Recurrence Relations for Sequences Like $\{F_{F_n}\}$," The Fibonacci Quarterly, April, 1967, Vol. 5, No. 2, pp. 129-136:

1. Replace "n" by "F" in the first line of the third paragraph on p. 129.
2. Replace the equations of (7') on page 132 by

$$2X_{n+2} = X_{n+1}Y_n + X_nY_{n+1}$$
$$2Y_{n+2} = (r - s)^2X_{n+1}X_n + Y_{n+1}Y_n\ .$$

3. Replace the "aj" in the first line of p. 134 by "a_j".
4. Replace the minus sign in the line preceding (15) on p. 134 by a plus.
5. Delete the first "4E" on the first line of page 136.

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Please also correct "A Shift Formula for Recurrence Relations of Order m," The Fibonacci Quarterly, December 1967, Vol. 5, No. 5, pp. 461-465, by replacing the "p_m" in the sum on the last line of p. 462 by "p_{m-i}".

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Please make the following correction in

" The Fibonacci Quarterly, November, 1967, Vol. 5, No. 4, p. 370:

In the fourth line from the bottom, replace "difference of each pair" with "differences of the pairs."

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