RECREATIONAL MATHEMATICS

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This column, hopefully, will serve the need for mathematical relaxation and make the reader look again at the other articles in the Fibonacci Quarterly with a mind more receptive to the fascination of mathematics. Actually, readers of this Journal are already inclined this way since this Journal is devoted to the study of one of the most fascinating series of numbers ever discovered.

Numbers, Fibonacci or otherwise, will not always be touched upon — mathematics, after all, is more than that. I look forward to comments and contributions from readers.

DIGITAL DIVERSIONS

Express the Fibonacci numbers using the ten digits once only and in order and only the common mathematical operations and symbols. Try to avoid expressions included in brackets indicating the nearest whole integer. You should be able to extend the list below. It would be interesting to determine the largest possible Fibonacci number so expressible, or to see in how many different ways a given number can be expressed.

$$F_1 = F_2 = 1 = 0 - 1 + 2 - 3 + 4 - 5 - 6 - 7 + 8 + 9$$
$$F_3 = 2 = 0 + (1)(2) - 3 + 4 - 5 - 6 - 7 + 8 + 9$$
$$F_4 = 3 = 0 - 1 + 2 - 3 - 4 + 5 - 6 - 7 + 8 + 9$$
$$F_5 = 5 = 0 + 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9$$
$$F_6 = 8 = 0 + (1)(2) + 3 + 4 - 5 - 6 - 7 + 8 + 9$$
$$F_7 = 13 = 0 + 1 + 2 + 3 - 4 - 5 + 6 - 7 + 8 + 9$$
$$F_8 = 21 = 0 - 1 + 2 + 3 + 4 - 5 - 6 + 7 + 8 + 9$$
$$F_9 = 34 = 0 + (1)(2) + 3 + 4 - 5 + 6 + 7 + 8 + 9$$

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$F_{10} = 55 = 0 + 12 + 34 + 5 - 6 - 7 + 8 + 9$

$F_{11} = 89 = 0 + 1 + 2 + 34 + 56 + 7 - 8 - \sqrt{9}$

$F_{12} = 144 = 0 + 1 + 2 + 3 + 4 + 5! + 6 + 7 - 8 + 9$

A DUDENEY PROBLEM

Henry Ernest Dudeney (1857-1930), one of England's foremost puzzlists once posed the following problem: "It will be found that 32, 547, 891 multiplied by 6 (thus using all the nine digits once, and once only) gives the product 195, 287, 346 (also containing all the nine digits once, and once only). Can you find another number to be multiplied by 6 under the same conditions? Remember that the nine digits must appear once, and once only, in the numbers multiplied and in the product."

Dudeney, in [1], included this problem, with the answer (6)(94, 857, 312) = 569, 143, 872. Martin Gardner, in editing this book, added two solutions supplied by Victor Meally: (6)(89, 745, 321) = 538,471,926 and (6)(98, 745, 231) = 592, 471, 386. With the help of a table constructed in 1963 by Harry L. Nelson of Livermore, California, I found that there are actually 87 solutions to this problem. These are listed in Table 1.

An obvious variation on Dudeney's problem is to ask the same question, but include zero as the tenth digit. There are 174 10-digit solutions derivable from Table 1 by simply appending a zero to one of the factors and the product. Examination of the table discloses many additional 10-digit solutions with the zero not at a terminal position. For example, the first three listed each yield two additional 10-digit solutions:

(6)(201, 578, 943) = 1, 209, 473, 658
(6)(215, 078, 943) = 1, 290, 473, 658
(6)(230, 158, 794) = 1, 380, 952, 764
(6)(231, 508, 794) = 1, 389, 052, 764
(6)(245, 098, 731) = 1, 470, 592, 386
(6)(245, 987, 301) = 1, 475, 923, 806

I leave it to the reader to find the other 10-digit solutions derivable from the table. However, I feel sure there may be other 10-digit solutions to the

Table 1

Solutions to Dudeney's Nine-Digit Problem

6 x 21578943 = 129473658	6 x 42985731 = 257914386	6 x 73195428 = 439172568
6 x 23158794 = 138952764	6 x 43152789 = 258916734	6 x 78195423 = 469172538
6 x 24598731 = 147592386	6 x 43195728 = 259174368	6 x 78219543 = 469317258
6 x 24958731 = 149752386	6 x 43219578 = 259317468	6 x 78549231 = 471295386
6 x 27548913 = 165293478	6 x 43271958 = 259631748	6 x 78942153 = 473652918
6 x 27891543 = 167349258	6 x 45719283 = 274315698	6 x 78943152 = 473658912
6 x 27893154 = 167358924	6 x 45719328 = 274315968	6 x 79854231 = 479125386
6 x 28731594 = 172389564	6 x 45728193 = 274369158	6 x 81954273 = 491725638
6 x 28943157 = 173658942	6 x 45731928 = 274391568	6 x 82719543 = 496317258
6 x 29415873 = 176495238	6 x 45781923 = 274691538	6 x 85473291 = 512839746
6 x 31275489 = 187652934	6 x 45782193 = 274693158	6 x 85491273 = 512947638
6 x 31542789 = 189256734	6 x 45819273 = 274915638	6 x 87249531 = 523497186
6 x 31578942 = 189473652	6 x 45827193 = 274963158	6 x 87294153 = 523764918
6 x 31587294 = 189523764	6 x 47328591 = 283971546	6 x 87315294 = 523891764
6 x 32458971 = 194753826	6 x 47532891 = 285197346	6 x 87495231 = 524971386
6 x 32547891 = 195287346	6 x 48572931 = 291437586	6 x 87941523 = 527649138
6 x 32714589 = 196287534	6 x 48579231 = 291475386	6 x 89145327 = 534871962
6 x 32897541 = 197385246	6 x 48591273 = 291547638	6 x 89532471 = 537194826
6 x 41527893 = 249167358	6 x 48912753 = 293476518	6 x 89532714 = 537196284
6 x 41957283 = 251743698	6 x 49285731 = 295714386	6 x 89745321 = 538471926
6 x 41957328 = 251743968	6 x 52487931 = 314927586	6 x 94152873 = 564917238
6 x 41957823 = 251746938	6 x 52874931 = 317249586	6 x 94857123 = 569142738
6 x 41958273 = 251749638	6 x 52987431 = 317924586	6 x 94857213 = 569143278
6 x 42195783 = 253174698	6 x 71528943 = 429173658	6 x 94857312 = 569143872
6 x 42319578 = 253917468	6 x 71954283 = 431725698	6 x 95248731 = 571492386
6 x 42719583 = 256317498	6 x 71954328 = 431725968	6 x 97328541 = 583971246
6 x 42731958 = 256391748	6 x 72819543 = 436917258	6 x 98541273 = 591247638
6 x 42789153 = 256734918	6 x 72854931 = 437129586	6 x 98724531 = 592347186
6 x 42819573 = 256917438	6 x 72985431 = 437912586	6 x 98745231 = 592471386

problem that are not derivable from the table. At this point, I can only hope that someone will use a computer to find all the 10-digit solutions to this particular problem.

The Nelson table mentioned previously was constructed in answer to a query I had made concerning the solution to the problem: What two or more factors containing the nine (or ten) digits once only yield a product containing the nine (or ten) digits once only? Nelson's computer-calculated table listed all 2,624 solutions to the 9-digit case (zero excluded). (See [4] for a discussion of this table.) The work involved in finding all the 10-digit solutions was not done. Included in Nelson's table are all the solutions to two other variations on Dudeney's problem. Substitute 3 or 9 in place of 6 as a factor. There are 335 solutions to the (3)(A) = B variation and 144 solutions to the (9)(C) = B variation, where A contains eight distinct digits (excluding zero and 3), B contains nine distinct digits (zero excluded,) and C contains eight distinct digits (zero and 9 excluded). Interested readers may obtain one free copy of these (3)(A) = B and (9)(C) = B tables simply by requesting them. Please, only one copy. If you want more, include at least five cents postage for every two copies.

ANOTHER DUDENEY PROBLEM

Dudeney once asked what numbers have cube roots equal to the sum of their digits. Excluding the trivial $1^3 = 1$, Dudeney [2] gave the five solutions: 512, 4913, 5832, 17576, and 19683. That is, $512 = (5 - 1 + 2)^3 = 8^3$; $(4 + 9 + 1 + 3)^3 = 17^3$; and so on.

Some years ago, I asked T. Charles Jones, then a student at Davidson College in Davidson, North Carolina, to run a computer search for solutions to this problem for n^{th} roots to $n = 101$. (The requests I sometimes put to people are not often trivial!) Elsewhere [4] I've shown how one might systematically search for these rather interesting numbers. These numbers — which, by the way, lack a precise name* — can be written as

$$N = abcd\cdots = (a + b + c + d + \cdots)^n = P^n .$$

*In [4] numbers which are representable, in some way, by mathematically manipulating their digits are called narcissistic. Closely related to the above numbers are those which are equal to the sum of the n^{th} powers of their digits; e.g., $153 = 1^3 + 5^3 + 3^3$. Such numbers are called Perfect Digital Invariants (PDI's) by Max Rumney of England, who has studied them extensively [5].

where abcd... represents the digits of N, and $(a + b + c + d + \cdots)$ represents the sum of the digits of N. Table 2 lists the 432 values of P which, when related to the n^{th} power, yield an N, the sum of whose digits is equal to P.

One of the interesting aspects of this problem is that there is at least one representative for every n from $n = 2$ to $n = 101$, with a maximum number (13) of representations at $n = 25$. Trivial representations such as $1^n = 1$ are not listed. The greatest number of times that a given P occurs is five: $P^n = N$ for $P = 90$ and $n = 19, 20, 21, 22$, and 28. The fully printed-out numbers total 19 computer sheets, but readers might be interested in seeing several of the larger examples.

P^n	=	N	(Sum of the digits in N is equal to P)						
181^{16}	=	13	26958	06363	75768	00539	94757	97274	10881
187^{16}	=	22	35968	62152	63449	25885	78257	92399	57441
499^{43}	=	10	43094	03484	75692	24451	60376	10004	44524
			27960	69557	10166	43340	61295	76132	73343
			99292	16069	53092	75509	14486	32354	72591
			73992	71499					
999^{75}	=	92770	86733	90001	46643	21616	99937	58761	27716
		93772	92872	78273	34425	52852	00275	13591	27714
		15647	08297	24430	57342	37029	14944	28952	64407
		21199	26192	76548	53218	72362	23108	52440	33783
		01874	09642	00691	32958	96038	80592	97398	10590
		35077	08174	61752	22250	74999			

The largest known number of this type is 1468^{101} which contains 320 digits — whose sum is 1468.

I found, quite by accident, one example of $P^n = N$ where the sum of the digits in N is equal to n:

$$2^{70} = 1,180,591,620,717,411,303,424.$$

Are there any more of this type?

Table 2

$N = P^n$, Where the Sum of the Digits in N equal P

n	P
2	9
3	8, 17, 18, 26, 27
4	7, 22, 25, 28, 36
5	28, 35, 36, 46
6	18, 45, 54, 64
7	18, 27, 31, 34, 43, 53, 58, 68
8	46, 54, 63
9	54, 71, 81
10	82, 85, 94, 97, 106
11	98, 107, 108, 117
12	108
13	20, 40, 86, 103, 104, 106, 107, 126, 134, 135, 146
14	91, 118, 127, 135, 154
15	107, 134, 136, 152, 154, 172, 199
16	133, 142, 163, 169, 181, 187
17	80, 143, 171, 216
18	172, 181
19	80, 90, 155, 157, 171, 173, 181, 189, 207
20	90, 181, 207
21	90, 199, 225
22	90, 169, 193, 217, 225, 234, 256
23	234, 244, 271
24	252, 262, 288
25	140, 211, 221, 236, 256, 257, 261, 277, 295, 296, 298, 299, 337
26	306, 307, 316, 324
27	305, 307
28	90, 160, 265, 292, 301, 328
29	305, 314, 325, 332, 341
30	396
31	170, 331, 338, 346, 356, 364, 367, 386, 387, 443
32	388
33	170, 352, 359, 378, 406, 422, 423
34	387, 412, 463
35	378, 388, 414, 451, 477
36	388, 424
37	414, 421, 422, 433, 469, 477, 485, 495
38	468, 469
39	449, 523
40	250, 441, 468, 486, 495, 502
41	432
42	280, 487, 523, 531
43	461, 499, 508, 511, 526, 532, 542, 548, 572
44	280, 523, 549, 576, 603
45	360, 503, 523
46	360, 478, 514, 522, 544, 558, 574, 592
47	350, 559, 567, 575, 595, 603, 666
48	370, 513, 631, 667
49	270, 290, 340, 350, 360, 533, 589, 637, 648, 661, 695

Table 2

(Continued from P. 65)

n	P
50	685
51	360, 666, 685
52	625, 688, 736, 739
53	648, 683, 703, 746
54	370, 603, 657, 667, 739
55	677, 683
56	684
57	370, 460, 719, 748, 793, 802
58	667, 721, 754
59	370, 440, 693, 845
60	694, 784, 792, 793
61	440, 490, 758, 815, 833
62	855, 865
63	793, 827, 836, 846
64	430, 829, 871
65	818, 856, 891, 928
66	837, 864, 927
67	450, 859, 865, 866, 869, 874 926, 934
68	837
69	540, 936, 962, 963, 1016
70	540, 882, 909
71	917, 991
72	901, 1062
73	853, 882, 928, 1006, 1015
74	936, 1008, 1009, 1018
75	630, 964, 999, 1016, 1053
76	1044, 1075, 1093
77	1061, 1062, 1088
78	964, 1117, 1126, 1134
79	610, 1031, 1043, 1054, 1064, 1091, 1108, 1133
80	1044, 1071, 1134, 1144
81	1062, 1196
82	1048, 1111, 1134, 1231
83	730, 1115, 1151, 1207
84	1188
85	1051, 1103, 1165, 1183, 1277
86	1134, 1225
87	1187, 1216, 1224, 1232, 1278, 1288
88	730, 1084, 1147, 1183, 1186, 1206
89	1151, 1232, 1358
90	1306, 1422
91	720, 1208, 1233, 1253, 1261, 1278
92	720, 1296, 1359
93	810, 820, 1396
94	1285, 1287, 1303, 1327, 1332, 1339, 1341, 1444
95	820, 1323, 1342, 1351, 1385
96	1387
97	1237, 1322, 1324, 1361, 1367, 1397, 1442
98	1359
99	1322, 1403, 1405, 1441
100	1363, 1378, 1408, 1414, 1489
101	1423, 1468.

A FIBONACCI VARIATION

Everyone tries his hand at variations on the Fibonacci theme. Mark Feinberg [3] has given us the Tribonacci and Tetranacci numbers, for example, where the terms of the series are the sums of the previous three or four terms, respectively. I hate to be excluded, so here's mine. The results turned out to be interesting, if not exactly stupendous. Form the ${}_nF$ series in each term is the sum of the NEXT TWO terms, and which starts with ${}_0F = 0$ and ${}_1F = 1$. The series, then, is

$$0,\ 1,\ -1,\ 2,\ -3,\ 5,\ -8,\ 13,\ -21,\ 34,\ -55,\ \text{etc.}$$

Note that if n is zero or even, the ${}_nF = -F_n$; if n is odd, then ${}_nF = F_n$. Can anyone do anything with this series?

SOME FIBONACCI QUERIES

What Fibonacci numbers are integral multiples of the sums of their digits? For example,

$$F_8 = 21,\quad 2+1 = 3,\quad \text{and}\ (3)(7) = 21;\ F_{12} = 144,\ 1+4+4 = 9,\ \text{and}\ (9)(16) = 144;\ F_{18} = 2584,\ 2+5+8 = 19,\ \text{and}\ (19)(136) = 2584.$$

I'm sure there are more. Are there an infinite number of them? Are they a function of n?

Somewhat related to the above is the problem of finding $F_n = N$, such that $N = nk$, where k is a positive integer. For example,

$$F_1 = 1;\ F_5 = 5;\ F_{12} = 144\ (\text{here}\ k = 12);\ F_{25} = 75025\ (\text{here}\ k = 3001)$$

Is there a formula relating these?

REFERENCES

1. Dudeney, Henry Ernest, 536 Puzzles and Curious Problems, edited by Martin Gardner, Charles Scribner's Sons, N. Y., 1967, pp. 41 and 257.

2. Ibid, pages 36-37 and 253.

3. Feinberg, Mark, "Fibonacci-Tribonacci," Fibonacci Quarterly, Vol. 1, No. 3 (October 1963), pp. 71-74.

4. Madachy, Joseph S., Mathematics on Vacation, Charles Scribner's Sons, N. Y., 1966, Chapter 6.

5. Rumney, Max, "Digital Invariants," Recreational Mathematics Magazine. No. 12 (December 1962), pp. 6-8.

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