A SEQUENCE OF POWER FORMULAS

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Starting with the familiar formulas

(1) $$F_{n+1} = F_n + F_{n-1}$$

(2) $$F^2_{n+1} = 2F^2_n + 2F^2_{n-1} - F^2_{n-2}$$

in which a power of a Fibonacci number is expressed as a linear combination of the same power of successive Fibonacci numbers one is led to seek additional formulas of this type.

For the third degree one can start with

$$F^3_{n+1} = F^3_n + 3F^2_n F_{n-1} + 3F_n F^2_{n-1} + F^3_{n-1}$$

$$F^3_{n-2} = F^3_n - 3F^2_n F_{n-1} + 3F_n F^2_{n-1} - F^3_{n-1}$$

$$F^3_{n-3} = -F^3_n + 6F^2_n F_{n-1} - 12F_n F^2_{n-1} + 8F^3_{n-1}$$

which result from cubing familiar linear relations. To arrive at the desired power relation, it is necessary to eliminate the terms that are not simple powers. Multiplying the first relation by a, the second by b and adding the result to the third yields the following relations for this elimination.

$$a - b + 2 = 0$$
$$a + b - 4 = 0$$

from which $b = 3$ and $a = 1$. This gives the desired relation of the third degree:

(3) $$F^3_{n+1} = 3F^3_n + 6F^3_{n-1} - 3F^3_{n-2} - F^3_{n-3}$$

This method can be pursued making use of coefficients without writing out complete expressions. For the fourth degree this gives a table:

F^4_{n+1}	1	4	6	4	1
F^4_{n-2}	1	-4	6	-4	1
F^4_{n-3}	1	-8	24	-32	16
F^4_{n-4}	16	-96	216	-216	81

This table leads to the following equations for eliminating the middle terms.

$$a + b - 2c - 24 = 0$$

$$a + b + 4c + 36 = 0$$

$$a - b - 8c - 54 = 0$$

from which $a = 1$, $b = 15$, $c = 5$, and $d = -1$. This leads to the relation:

$$F^4_{n+1} = 5F^4_n + 15F^4_{n-1} - 15F^4_{n-2} - 5F^4_{n-3} - F^4_{n-4} \tag{4}$$

Fifth and sixth degree relations are:

$$F^5_{n+1} = 8F^5_n + 40F^5_{n-1} - 60F^5_{n-2} - 40F^5_{n-3} + 8F^5_{n-4} + F^5_{n-5} \tag{5}$$

$$F^6_{n+1} = 13F^6_n + 104F^6_{n-1} - 260F^6_{n-2} - 260F^6_{n-3} + 104F^6_{n-4} + 13F^6_{n-5} - F^6_{n-6} \tag{6}$$

Since the algebra at this point was becoming laborious, the coefficients were set up in tabular form for the purpose of discovering a pattern. The heading is the degree; the numbers below are the successive coefficients of the terms on the right-hand side of the relation.

1	2	3	4	5	6
1	2	3	5	8	13
1	2	6	15	40	104
	-1	-3	-15	-60	-260
		-1	-5	-40	-260
			1	8	104
				1	13
					-1

It was observed that one column can be obtained from the previous column by multiplying by a Fibonacci number and dividing successive products by certain Fibonacci numbers in reverse order. Thus to go from the column headed 4 to the column headed 5, multiply each quantity in the 4 column by 8 and divide successive products by 5, 3, 2, 1, 1 respectively. To go from column 5 to column 6, multiply each quantity in column 5 by 13 and divide by 8, 5, 3, 2, 1, 1 respectively. The new elements in each column at the end are all 1's with a plus or minus sign, the order being two minuses, two etc.

With the aid of this empirical result, the table was continued to higher powers.

7	8	9	10	11	12	13
21	34	55	89	144	233	377
273	714	1870	4895	12816	33552	87841
-1092	-4641	-19635	-83215	-352440	-1493064	-6324552
-1820	-12376	-85085	-582505	-3994320	-27372840	-187628376
1092	12376	136136	1514513	16776144	186145312	2063912136
273	4641	85085	1514513	27261234	488605194	8771626578
-21	-714	-19635	-582505	-16776144	-488605194	-14169550626
-1	-34	-1870	-83215	-3994320	-186135312	-8771626578
	1	55	4895	352440	27372840	2063912136
		1	89	12816	1493064	187628376
			-1	-144	-33552	-6324552
				-1	-233	-87841
					1	377
						1

In each instance the coefficients were checked by applying the formula to one particular value of n.

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