CONJUGATE GENERALIZED FIBONACCI SEQUENCES

Charles H. King
San Jose, California

1. INTRODUCTION

The famous rabbit problem of Leonardo Fibonacci gives the sequence

(1) $$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ \cdots$$

where each element is the number of rabbits present in that time period. In 1634, Albert Girard discovered the law,

(2) $$U_{n+2} = U_{n+1} + U_n$$

for the sequence (1). Any pair of positive integers U_n and U_{n+1} substituted in (2) generates a sequence; if $U_1 = 1$ and $U_2 = 2$ the Fibonacci Sequence is generated and $U_1 = 1$, $U_2 = 3$ generates the Lucas Sequence.

Consider any pair (a_n, a_{n+1}) used in the generating function (2); if $a_n > a_{n+1}$ then a_n must be the first element of the sequence, because no positive integer added to a_n will make the sum smaller, and the same sequence may be generated by the pair (a_{n+1}, a_{n+2}) thus the restriction $a_n \leq a_{n+1}$ with no loss of generality. Continuing with the pair (a_n, a_{n+1}) thus restricted the generating function (2). $U_{n-1} = U_{n+1} - U_n$ shows that as n decreases the elements in the sequence become smaller and thus there must be a least positive element. In the case of the Fibonacci Sequence there are two equal least positive elements both one. Let a_1 be the least positive element for some sequence and a_2 the next element for increasing n then the pair (a_1, a_2) characterize the sequence.

If the pair (a_1, a_2) have a common factor, $(a_1, a_2) = k$, then all elements of the sequence have the same common factor and the sequence may be represented by another relatively prime pair $(a_1', a_2') = 1$ where,

$$a_1' = \frac{a_1}{k}, \qquad a_2' = \frac{a_2}{k} .$$

When the elements of the sequence generated by (a_1', a_2') are multiplied by k the original sequence is recovered. A sequence is a primitive sequence if $(a_1, a_2) = 1$. For any pair where $a_2 > 2$, then $2a_1 < a_2$. If $2a_1 \not< a_2$ then there would be some $a_0 > 0$ such that $a_0 + a_1 = a_2$ where $a_0 < a_1$ and a_1 would not be the least positive element. Thus any sequence generated by (2) may be defined by a positive pair of integers $(a_1, a_2) = 1$ and $2a_1 < a_2$. The exception, of course, is the Fibonacci Sequence.

2. DEFINITION OF CONJUGATES

From the pair (a_1, a_2) it has been shown that a_1 is the least positive element in the sequence that (a_1, a_2) defines, but there is also a number a_0 satisfying (2), $a_0 + a_1 = a_2$. For the Fibonacci Sequence $a_0 = 0$ and $a_0 = 2$ for the Lucas Sequence. If negative elements are allowed then there is an a_{-1} satisfying $a_{-1} + a_0 = a_1$ and $a_{-2}, a_{-3}, a_{-4}'''$ can be found. Thus there are values positive or negative for all a_{-n}.

The absolute values of the elements of the sequence formed by a_{-n} form a sequence which is called the conjugate sequence of the original positive sequence. The element a_0 is the zero$^{\text{th}}$ element of both sequences. If the elements of the conjugate sequence equal the elements of the original sequence (if $a_n = |a_{-n}|$) the sequence is called self-conjugate. The Fibonacci and Lucas Sequences are the only self-conjugate Fibonacci-Type sequences. Given a pair (a_1, a_2) defining a sequence, the pair (a_1^*, a_2^*) defining its conjugate sequence may be found by solving the equations,

$$a_1^* = a_2 - 2a, \quad a_2^* = 2a_2 - 3a \;, \tag{3}$$

These pairs (a_1, a_2) and (a_1^*, a_2^*) are called conjugate pairs. The conjugate sequences and pairs are illustrated in the table below.

Table 1

(a_1, a_2)	(a_1^*, a_2^*)	U_{-4}	U_{-3}	U_{-2}	U_{-1}	U_0	U_1	U_2	U_3	U_4
(1,1)	(1, 1)	-3	2	-1	1	0	1	1	2	3
(1,3)	(1,3)	7	-4	3	-1	2	1	3	4	7
(1,4)	(2,5)	2	-7	5	-2	3	1	4	5	9
(2,5)	(1,4)	9	-5	4	-1	3	2	5	7	12

3. THE CHARACTERISTIC NUMBER D

With any pair (a_1, a_2) there is a number D determined by the equation

$$D = a_2^2 - a_1(a_1 + a_2) \tag{4}$$

A table of D less than 1000 is given in [1]. For a pair (a_1, a_2) determining a D there is associated another conjugate pair (except (1, 1) and (1, 3) that also determine the same D. These pairs are conjugate pairs as defined above. All prime D greater than 5 have the form $P_i = (10k \pm 1)$ and all composite D are in the form,

$$D = 5^{\alpha_0} P_1^{\alpha_1} P_2^{\alpha_2} P_3^{\alpha_3} \cdots P_n^{\alpha_n} \tag{5}$$

all P_i are prime D of the form $(10\,k \pm 1)$.

There may be more than one set of conjugate pairs that give a D. For D less than 1000, the number of conjugate pairs associated with any D may be found from the factorization of D as follows,

$$D = 5^{\alpha_0} P_1^{\alpha_1} P_2^{\alpha_2} \cdots P_n^{\alpha_n} \tag{6}$$

1. One pair $\begin{cases} \alpha_0 \neq 0, & \text{or} \\ \text{any } \alpha_i = 2 \end{cases}$
2. Two pair, $\alpha_0 = 0$, all $d_i < 2$, and at least 2 distinct P_i.

Any two adjacent elements U_n, U_{n+1} from any primitive sequence substituted in the generating function,

$$U_n \cdot U_{n+1} + (U_n + U_{n+1})^2 \tag{7}$$

will give a number from the set of D_3. From the Fibonacci Sequence

$[1, 1] \longrightarrow 5$, $[1, 2] \longrightarrow 11$, $[2, 3] \longrightarrow 31$, $[3, 5] \longrightarrow 79$.

When a D is determined by (7) a conjugate pair that determine the same D by (4) may be found from the adjacent elements used in (7). Let (F_n, F_{n+1}) be the adjacent elements that give D_n from (7) then a conjugate pair that determine the same D_n from (4) is

$$(F_n,\ F_n + F_{n+2}),\ (F_{n+1},\ F_{n+3}) \tag{8}$$

For example the Fibonacci Sequence 1, 1, 2, 3, 5, 8, 13, $\cdots$ take the adjacent pair [2, 3] in (7); this gives D = 31. The conjugate pair that give D = 31 from (4) is (2, 7), (3, 8) and from (8) where

$$F_n = 2, \quad F_{n+1} = 3, \quad F_{n+2} = 5, \quad F_{n+8} = 8 \ ,$$

the pair is also (2, 7) (3, 8).

If (7) is used to generate D's by using all adjacent elements in all primitive sequences then all D will be generated and each will appear the number of times equal to the number of conjugate pairs associated with it.

REFERENCE

1. Brother U. Alfred, "On the Ordering of Fibonacci Sequences," The Fibonacci Quarterly, 1 (4), 1963, pp. 43-46.

* * * * *

ERRATA

Please make the following corrections on the article, "A Primer for the Fibonacci Numbers, Part VI," appearing in the December, 1967, Vol. 5, No. 5, issue of the Fibonacci Quarterly, pp. 445-460:

p. 446: In the fifth line from the bottom, replace "indeterminant" with "indeterminate."

p. 452: In the line before relation (3.5), insert "of" before x^n.

p. 455: In the ninth line from the bottom, replace (3.3) by (3.4).

* * * * *

Please make the following correction in the Vol. Index, Vol. 5, No. 5, December, 1967, issue of the Fibonacci Quarterly: Change S. D. Mohanty to S. G. Mohanty on p. 495 .

* * * * *