PYTHAGOREAN TRIADS OF THE FORM X, X+1, Z DESCRIBED BY RECURRENCE SEQUENCES

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The term Pythagorean Triples or Triads is applied to those integers which describe all right triangles with integral sides. The sub-class which is the subject of this paper, is restricted to those of sides x, $x + 1$, $\sqrt{2x^2 + 2x + 1}$. It is obvious that the smallest such triangle has sides 3, 4, 5. The problem is to find a general method of sequential progress through the family of all such triangles. In the course of this development, and consequent to a solution of Pell's equation, it is shown that these triangles bear a curious relationship to a series which, with the exception of a single coefficient, is identical with the Fibonacci series.

It can be shown that in a right triangle $x^2 + y^2 = z^2$, primitive solutions are given by integers a, b such that $x = a^2 - b^2$, $y = 2ab$ and $z = a^2 + b^2$ where $a > b$, and (a, b) are relatively prime. This paper will be concerned with triangles in which $y = x \pm 1$, or $x^2 + (x \pm 1)^2 = z^2$, the primitive solutions of which also take this form.

A. If x is odd and

$$x = a^2 - b^2 \quad \text{and} \quad x + 1 = 2ab,$$

then

$$-1 = a^2 - 2ab - b^2$$

$$-1 = a^2 - 2ab - b^2 + b^2 - b^2$$

$$-1 = a^2 - 2ab + b^2 - 2b^2$$

$$-1 = (a - b)^2 - 2b^2$$

B. If x is even and

$x = 2ab$ and $x + 1 = a^2 - b^2$ (Note: In A, x was odd and in B, x is even in order to account for all possibilities.)

then

$$+1 = a^2 - 2ab - b^2$$

$$+1 = (a - b)^2 - 2b^2$$

Let $p = a - b$ and $q = b$, then by A and B above

(1) $$\pm 1 = p^2 - 2q^2 .$$

Equation (1) is an example of Pell's equation. By inspection, the smallest integral solution greater than zero of this equation is $p = 1$, $q = 1$.

Equation (1) can be factored into

$$(p - q\sqrt{2})(p + q\sqrt{2}) = \pm 1$$

which, when raised to the n^{th} power, becomes

$$(p - q\sqrt{2})^n (p + q\sqrt{2})^n = \pm 1$$

Specifically

$$(1 - \sqrt{2})^n (1 + \sqrt{2})^n = \pm 1$$

since $p = 1$, $q = 1$ is a solution of equation (1).

Now let

(2) $$p_n + q_n\sqrt{2} = (1 + \sqrt{2})^n$$

then

(3) $$p_n - q_n\sqrt{2} = (1 - \sqrt{2})^n$$

Then, by solving these simultaneous equations,

$$p_n = 1/2\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right] \tag{4}$$

$$q_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right] \tag{5}$$

Since $p = 1$, $q = 1$ is the smallest solution of equation (1), then the general solution is given by (2) or (3) above and, therefore, by (4) and (5). (This can be found in most texts on Number Theory.)

Adding equations (4), (5)

$$p_n = 1/2\left[(1+\sqrt{2})^n + (1-\sqrt{2})^n\right] \tag{4a}$$

$$q_n = \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^n - (1-\sqrt{2})^n\right] \tag{5a}$$

$$p_n + q_n = \frac{1}{2\sqrt{2}}\left[\sqrt{2}(1+\sqrt{2})^n + \sqrt{2}(1-\sqrt{2})^n + (1+\sqrt{2})^n - (1-\sqrt{2})^n\right]$$

$$= \frac{1}{2\sqrt{2}}\left[(\sqrt{2}+1)(1+\sqrt{2})^n - (1-\sqrt{2})(1-\sqrt{2})^n\right]$$

$$= \frac{1}{2\sqrt{2}}\left[(1+\sqrt{2})^{n+1} - (1-\sqrt{2})^{n+1}\right]$$

$$p_n + q_n = q_{n+1} \tag{6}$$

Since $p_n = a - b$ and $q_n = b$, then

$$a = p_n + q_n$$

or

$$a = q_{n+i}$$

and, of course,

$$b = q_n$$

Equation (2) can be rewritten

$$
\begin{aligned}
p_{n+1} + q_{n+1}\sqrt{2} &= (1 + \sqrt{2})^{n+1} \\
&= (1 + \sqrt{2})^{n}(1 + \sqrt{2}) \\
&= (p_n + q_n\sqrt{2})\ (1 + \sqrt{2}) \\
&= p_n + p_n\sqrt{2} + q_n\sqrt{2} + 2q_n \\
&= (p_n + 2q_n) + \sqrt{2}(p_n + q_n)
\end{aligned}
$$

But

$$p_n + q_n = q_{n+1}$$

(7) $$\therefore\ p_{n+1} = p_n + 2q_n$$

Rewriting equations (7), (6) and subtracting,

(7.a) $$p_{n-1} = p_{n-2} + 2q_{n-2}$$

(6.a) $$q_{n-1} = p_{n-2} + q_{n-2}$$

(8) $$p_{n-1} = q_{n-1} + q_{n-2}$$

Now rewriting equation (6)

(6.b) $$q_n = p_{n-1} + q_{n-1}$$

Substitute equation (8)

(9)
$$
\begin{aligned}
q_n &= q_{n-1} + q_{n-2} + q_{n-1} \\
q_n &= 2q_{n-1} + q_{n-2}
\end{aligned}
$$

In both A and B above, the term 2ab was used, once for x and once for x + 1. If p and q satisfy $p^2 - 2q^2 = -1$, then $x + 1 = 2ab$. If p and q satisfy $p^2 - 2q^2 = +1$, then $x = 2ab$. Equations (2) and (3) state that the only way for the negative portion of equation (1) to be satisfied is for $(1 - \sqrt{2})^n$ to be negative. If $(1 - \sqrt{2})^n$ is negative, then $x + 1 = 2ab$; if $(1 - \sqrt{2})^n$ is positive, then $x = 2ab$. Since $(1 - \sqrt{2})$ is a negative term $(\sqrt{2} > 1)$, $(1 - \sqrt{2})^n$ is positive when n is even and negative when n is odd. Now the formula for one side of the triangle becomes

$$(10) \qquad 2q_n q_{n+1} = \begin{cases} x & \text{for even values of } n \\ x + 1 & \text{for odd values of } n \end{cases}$$

We have now developed a recurrence relationship for the q terms in relation to previous q terms (equation 9).

Except for the coefficient 2 of q_{n-1}, this is the Fibonacci Series. Note that in this same manner the expression $p_n = 2p_{n-1} + p_{n-2}$ can also be proved.

Until now nothing has been formulated concerning the hypotenuse or z term of the Pythagorean Triple. Since squaring and taking the root of very large numbers is difficult, it would be advantageous to have a recursive formula for the z terms. We propose to prove that

$$(11) \qquad z_n = q_{2n+1}$$

is such a formula. Then any Pythagorean Triad of the form x, x + 1, z can be found recursively by using equations (9), (10), and (11). Further, by use of equation (6), any two consecutive q terms can be found and the sequence proceeds from there. See Appendix A. Proof for equation (11) follows.

From A and B above, two conditions are possible, either $x = a^2 - b^2$ and $x + 1 = 2ab$ or $x = 2ab$ and $x + 1 = a^2 - b^2$. In either case,

$$x^2 + (x + 1)^2 = (a^2 - b^2)^2 + (2ab)^2 .$$

As stated before,

$$2ab = 2q_n q_{n+1}$$

for the n^{th} triad. Also,

$$a^2 - b^2 = q_{n+1}^2 - q_n^2$$

since

$$a = q_{n+1} \text{ and } b = q_n$$

Then,

$$x^2 + (x + 1)^2 = \left(q_{n+1}^2 - q_n^2\right)^2 + \left(2q_n q_{n+1}\right)^2$$

$$= q_{n+1}^4 - 2q_n^2 q_{n+1}^2 + q_n^4 + 4q_n^2 q_{n+1}^2$$

$$= q_{n+1}^4 + 2q_n^2 q_{n+1}^2 + q_n^4$$

$$= \left(q_{n+1}^2 + q_n^2\right)^2$$

$$\sqrt{x^2 + (x + 1)^2} = z_n = q_{n+1}^2 + q_n^2$$

To prove equation (11) all that remains is to prove that

$$q_{2n+1} = q_{n+1}^2 + q_n^2$$

To do this we will prove by induction on k that

$$q_{2n+1} = q_{k+2}\, q_{2n-k} + q_{k+1}\, q_{2n-(k+1)} \;.$$

If $k = 0$

$$q_{2n+1} = 2q_{2n} + q_{2n-1}$$

$$q_{2n} = 2q_{2n-1} + q_{2n-2}$$

$$q_{2n+1} = 2\left[2q_{2n-1} + q_{2n-2}\right] + q_{2n-1}$$

If $k = 1$

$$q_{2n+1} = 5q_{2n-1} + 2q_{2n-2}$$

Notice now that q_{2n+1} is represented in terms of

$$(q_3 = 5,\ q_{2n-1}),\ (q_2 = 2, \text{ and } q_{2n-2}).$$

Assume that the k^{th} relationship is of the form

$$q_{2n+1} = q_{k+2}\,q_{2n-k} + q_{k+1}\,q_{2n-(k+1)}$$

Certainly the first relationship is true as we have just shown. Assume the k^{th} relationship is true. Then,

$$q_{2n+1} = q_{k+2}\,q_{2n-k} + q_{k+1}\,q_{2n-(k+1)}$$

From equation (9) we know

$$q_{2n-k} = 2q_{2n-k-1} + q_{2n-k-2}$$

Then

$$q_{2n+1} = q_{k+2}\left[2q_{2n-k-1} + q_{2n-k-2}\right] + q_{k+1}q_{2n-k-1}$$

$$q_{2n+1} = 2q_{k+2}\,q_{2n-k-1} + q_{k+2}\,q_{2n-k-2} + q_{k+1}\,q_{2n-k-1}$$

$$q_{2n+1} = q_{2n-k-1}\left[2q_{k+2} + q_{k+1}\right] + q_{k+2}\,q_{2n-k-2}$$

Since

$$2q_{k+2} + q_{k+1} = q_{k+3}\ ,$$

$$q_{2n+1} = q_{k+3}\,q_{2n-k-1} + q_{k+2}q_{2n-k-2}$$

This is the $(k + 1)^{st}$ relationship and this proves the general equation inductively. Specifically, when $k = n - 1$,

$$q_{2n+1} = q_{(n-1)+2}\, q_{2n-(n-1)} + q_{(n-1)+1}\, q_{2n-[(n-1)+1]}$$

$$q_{2n+1} = q_{n+1}\, q_{n+1} + q_n\, q_n$$

$$q_{2n+1} = q_{n+1}^2 + q_n^2$$

Then this completes the proof for equation (11).

APPENDIX A

n		q_n	$2q_nq_{n+1}$	$=\{$	x
1		1	4	x_1 =	3
2		2	20	x_2 =	20
3	$\underline{z_1}$ =	5	120	x_3 =	119
4		12	696	x_4 =	696
5	$\underline{z_2}$ =	29	4060	x_5 =	4059
6		70	23360	x_6 =	23360
7	$\underline{z_3}$ =	169	137904	x_7 =	137903
8		408	803760	x_8 =	803760
9	$\underline{z_4}$ =	985	4684660	x_9 =	4684659
10		2378	27304196	x_{10} =	27304196
11	$\underline{z_5}$ =	5741	159140520	x_{11} =	159140519
12		13860	927538920	x_{12} =	927538920
13	$\underline{z_6}$ =	33461	5406093004	x_{13} =	5406093003
14		80782	31509019100	x_{14} =	3150919100
15	$\underline{z_7}$ =	195025	183648021600	x_{15} =	183648021599
16		470832	1070387585472	x_{16} =	1070387585472
17	$\underline{z_8}$ =	1136689	6238626641380	x_{17} =	6238626641379
18		2744210	36361380737780	x_{18} =	36361380737780
19	$\underline{z_9}$ =	6625109	211929657785304	x_{19} =	211929657785303
20		15994428	1235216565974040	x_{20} =	1235216565974040
21	$\underline{z_{10}}$ =	38613965			
22		93222358			
23	$\underline{z_{11}}$ =	225058681			

APPENDIX A (Continued)

n			q_n	$2q_n\, q_{n+1}$	$= \{$	x
24			54339720			
25	$\underline{z_{12}}$	=	1311738121			
26			3166815962			
27	$\underline{z_{13}}$	=	7645370045			
28			18457556052			
29	$\underline{z_{14}}$	=	44560482149			
30			107578520350			
31	$\underline{z_{15}}$	=	259717522849			
32			527013566048			
33	$\underline{z_{16}}$	=	1513744654945			
34			4074502875938			
35	$\underline{z_{17}}$	=	9662750406821			
36			23400003689580			
37	$\underline{z_{18}}$	=	56462757785981			
38			136325519261542			
39	$\underline{z_{19}}$	=	329113796309065			
40			794553111879672			
41	$\underline{z_{20}}$	=	1918220020068409			

APPENDIX B

x	x + 1	z
3	4	5
20	21	29
119	120	169
696	697	985
4059	4060	5741
23360	23361	33461
137903	137904	195025
803760	803761	1136689
4684659	4684660	6625109
27304196	27304197	38613965
159140519	159140520	225058681
927538920	927538921	1311738121
5406093003	5406093004	7645370045
31509019100	31509019101	44560482149
183648021599	183648021600	259717522849
1070387585472	1070387585473	1513744654945
6238626641379	6238626641380	9662750406821
36361380737780	36361380737781	56462757785981
211929657785303	211929657785304	329113796309065
1235216565974040	1235216565974041	1918220020068409

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