GENERALIZED RABBITS FOR GENERALIZED FIBONACCI NUMBERS

V. E. Hoggatt, Jr.
San Jose State College, San Jose, Calif.

1. INTRODUCTION

The original Fibonacci number sequence arose from an academic rabbit production problem (see [1] and [5], pp. 2-3). In this paper we generalize the birth sequence pattern and determine the sequences of new arrivals and total population. We shall obtain the Fibonacci sequence in several different ways.

2. GENERAL BIRTH SEQUENCE

Consider a new-born pair of rabbits which produce a sequence of litters. Let the number of rabbit pairs in the n^{th} litter, which is delivered at the n^{th} time point, be B_n. Assume that each offspring pair also breeds in the same manner. Clearly $B_0 = 0$, and the B_n are nonnegative integers for $n \geq 1$.

The array (1) will aid us in our formalization. Let

$$B(x) = \sum_{n=0}^{\infty} B_n x^n \quad (B_0 = 0),$$

(1)
$$\begin{cases} R_0 = 1 \\ R_1 = B_1R_0 \\ R_2 = B_2R_0 + B_1R_1 \\ R_3 = B_3R_0 + B_2R_1 + B_1R_2 \\ \vdots \\ R_n = B_nR_0 + \cdots + B_1R_{n-1} \end{cases}$$

$$R(x) = \sum_{n=0}^{\infty} R_n x^n \quad (R_0 = 1), \qquad \text{and} \qquad T(x) = \sum_{n=0}^{\infty} T_n x^n \quad (T_0 = 1)$$

be the generating functions for the birth sequence, new arrival sequence, and total population sequence, respectively. Remembering that $B_0 = 0$, it is clear that

(2)
$$R_n = \sum_{j=0}^{n-1} B_{n-j}R_j = \sum_{j=0}^{n} B_{n-j}R_j \qquad (n \geq 1).$$

Noticing that (2) gives the incorrect result $R_0 = 0$ instead of the correct $R_0 = 1$, we have

$$R(x) - 1 = -1 + \sum_{j=0}^{\infty} R_j x^j = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n} B_{n-j}R_j\right) x^n .$$

$$= R(x)B(x) ,$$

so that

(3) $$R(x) = \frac{1}{1 - B(x)}$$

and

$$B(x) = \frac{R(x) - 1}{R(x)} .$$

Now

$$T_n = \sum_{j=0}^{n} R_j ,$$

so by summing the array (1) along the diagonals we can also write

$$T_n = 1 + \sum_{j=0}^{n-1} B_{n-j}T_j = 1 + \sum_{j=0}^{n} B_{n-j}T_j ,$$

since $B_0 = 0$. Thus

$$T(x) - \frac{1}{1 - x} = T(x)\,B(x) ,$$

so that

(4) $$T(x) = \frac{1}{(1 - x)(1 - B(x))} = \frac{R(x)}{1 - x} ,$$

$$B(x) = 1 - \frac{1}{(1 - x)T(x)} .$$

3. SOME INTERESTING SPECIAL CASES

The original Fibonacci rabbit problem has the birth sequence generating function as

$$B(x) = \frac{x^2}{1 - x} = \sum_{n=0}^{\infty} B_n x^n .$$

Here the new-born rabbit pairs mature for one period and then each pair gives birth to a new rabbit pair at each time point thereafter in their private times. Using (3),

$$R(x) = \frac{1}{1 - \frac{x^2}{1-x}} = \frac{1-x}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n-1}x^n ,$$

while equation (4) yields

$$T(x) = \frac{1}{(1-x)\left(1 - \frac{x^2}{1-x}\right)} = \frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+1}\, x^n .$$

Thus both the new arrival sequence and the total population sequence are Fibonacci sequences (see the chart in [1], p. 57).

We may also get Fibonacci sequences in other ways. Let

$$B(x) = \frac{x}{1-x^2} .$$

Then

$$R(x) = \frac{1}{1 - \frac{x}{1-x^2}} = \frac{1-x^2}{1-x-x^2} = 1 + \sum_{n=0}^{\infty} F_n x^n ,$$

and

$$T(x) = \frac{1+x}{1-x-x^2} = \sum_{n=0}^{\infty} F_{n+2}\, x^n .$$

In this birth sequence a rabbit pair produces and rests in alternate time periods.

If, on the other hand,

$$B(x) = x + x^2 ,$$

then

$$R(x) = \frac{1}{1-(x+x^2)} = \sum_{n=0}^{\infty} F_{n+1}\, x^n$$

and

$$T(x) = \frac{1}{(1-x)(1-x-x^2)} = \sum_{n=0}^{\infty} (F_{n+3} - 1)\, x^n .$$

Suppose instead

$$B(x) = \frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n .$$

Then

$$R(x) = \frac{(1-x)^2}{1-3x+x^2} = 1 + \sum_{n=0}^{\infty} F_{2n}\, x^n ,$$

and

$$T(x) = \frac{1-x}{1-3x+x^2} = \sum_{n=0}^{\infty} F_{2n+1}\, x^n .$$

Suppose we let the pair produce with a birth sequence which is the Fibonacci sequence. Then

$$B(x) = \frac{x}{1-x-x^2} ,$$

$$R(x) = \frac{1-x-x^2}{1-2x-x^2} = 1 + \sum_{n=0}^{\infty} C_n\, x^n ,$$

where $C_0 = 0$, $C_1 = 1$, and $C_{n+2} = 2C_{n+1} + C_n$ $(n \geq 0)$. We note that if

$$f_0(x) = 0, \quad f_1(x) = 1, \quad \text{and} \quad f_{n+2}(x) = xf_{n+1}(x) + f_n(x)$$

define the sequence of Fibonacci polynomials $\{f_n(x)\}$, then $C_n = f_n(2)$.

There is a typographical error in Weland [6]. When

$$B(x) = \sum_{n=2}^{\infty}\left(1 + \sum_{j=1}^{n} 6C_{j-1}\right)x^n ,$$

then

$$T(x) = \sum_{n=0}^{\infty} F_{n+1}^3\, x^n ,$$

where the C_n are the same as in the example immediately above.

4. SOME FURTHER FIBONACCI RESULTS

Since $F_{k-1} + F_{k+1} = L_k$, and every k^{th} Fibonacci number obeys the recurrence relations

$$y_{n+2} = L_k y_{n+1} - (-1)^k y_n ,$$

we can now give the following results. If

$$B(x) = \frac{F_{k+1}x - (-1)^k x^2}{1 - F_{k-1}x} = xF_{k+1} + x^2F_k^2 \sum_{j=0}^{\infty} F_{k-1}^j x^j ,$$

then

$$R(x) = \frac{1 - F_{k-1}x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn+1} x^n .$$

If

$$B(x) = \frac{F_{k-1}x - (-1)^k x^2}{1 - F_{k+1}x} = xF_{k-1} + x^2F_k^2 \sum_{j=0}^{\infty} F_{k+1}^j x^j ,$$

then

$$R(x) = \frac{1 - F_{k+1}x}{1 - L_k x + (-1)^k x^2} = \sum_{n=0}^{\infty} F_{kn-1} x^n .$$

If

$$B(x) = \frac{(F_{k+1} - 1)x + (F_{k-1} - (-1)^k)x^2}{(1 - x)(1 - F_{k-1}x)} ,$$

then

$$T(x) = \sum_{n=0}^{\infty} F_{kn+1} x^n .$$

If

$$B(x) = \frac{(F_{k-1} - 1)x + (F_{k+1} - (-1)^k)x^2}{(1 - x)(1 - F_{k+1}x)} ,$$

then

$$T(x) = \sum_{n=0}^{\infty} F_{kn-1} x^n .$$

We conclude this section with two final examples. Suppose the birth sequence is given by $B_0 = B_1 = 0$, $B_n = 2n - 1$ $(n \geq 2)$. Then we find

$$R(x) = \sum_{n=0}^{\infty} F_{n-1} F_{n+2} x^n$$

and

$$T(x) = \sum_{n=0}^{\infty} F_{n+1}^2 x^n .$$

We must not leave out the Lucas birth sequence. If

$$B(x) = \frac{x(2-x)}{1-x-x^2} = \sum_{n=0}^{\infty} L_n x^{n+1} ,$$

then

$$R(x) = \frac{1-x-x^2}{1-3x} = \sum_{n=0}^{\infty} R_n x^n$$

with

$$R_0 = 1, \quad R_1 = 2, \quad R_n = 5 \cdot 3^{n-2} \qquad n \geq 2$$

5. BIRTH SEQUENCES YIELDING GENERALIZED FIBONACCI NUMBERS

The generalized Fibonacci numbers $u(n; p, q)$ of Harris and Styles [2] have the generating function [4]

$$\frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n; p, q)\, x^n .$$

If

$$B(x) = \frac{x^{p+q}}{(1-x)^q} ,$$

then

$$T(x) = \frac{(1-x)^{q-1}}{(1-x)^q - x^{p+q}} \qquad (q \geq 1) .$$

We note that here the birth sequence $\{B_n\}$ starts with $p + q - 1$ zeros (maturing periods) and then proceeds down the $(q - 1)^{st}$ column of the left-justified Pascal's Triangle [4]. We note further that if

$$B(x) = x + \frac{x^{p+q}}{(1 - x)^{q-1}} ,$$

$$R(x) = \frac{(1 - x)^{q-1}}{(1 - x)^q - x^{p+q}} = \sum_{n=0}^{\infty} u(n;\ p,\ q)\ x^n \quad (q \geq 2) .$$

In this case $B_0 = 0$, $B_1 = 1$, $B_j = 0$ $(j = 2, \cdots, p + q - 1)$, and the sequence then proceeds down the $(q - 2)^{nd}$ column of the left-justified Pascal's Triangle. It was this interesting problem that inspired further research resulting in this paper.

6. A SECOND GENERALIZATION

Harris and Styles [3] gave a further generalization of the Fibonacci numbers by introducing the numbers

$$u(n;\ p,\ q,\ s) = \sum_{i=0}^{\left[\frac{n}{p+sq}\right]} \binom{\left[\frac{n - ip}{s}\right]}{iq} ,$$

where $[x]$ represents the greatest integer contained in x. It is shown in [4] that the generating function for these numbers is

$$\frac{(1 - x^s)^q \big/ (1 - x)}{(1 - x^s)^q - x^{p+sq}} = \sum_{n=0}^{\infty} u(n;\ p,\ q,\ s)\ x^n .$$

If

$$B(x) = \frac{x^{p+sq}}{(1 - x^s)^q} ,$$

then

$$T(x) = \frac{(1 - x^s)^q \Big/ (1 - x)}{(1 - x^s)^q - x^{p+sq}} \qquad (p + sq \geq 1) \qquad q \geq 1\,.$$

Therefore the birth sequence yielding u(n; p, q, s) as the total population sequence begins with $p + sq - 1$ zeros (maturing periods) and then has the entries of the $(q - 1)^{st}$ column of the left-justified Pascal's Triangle alternated with $s - 1$ zeros. The pair thus alternately produces and then rests for $s - 1$ periods after maturing for $p + sq - 1$ periods.

Note: Lucile Morton has now completed her San Jose State College Master's Thesis, "The Generalized Fibonacci Rabbit Problem," and the results will be written up in a paper to appear soon in the Fibonacci Quarterly.

REFERENCES

1. Brother U. Alfred, "Exploring Fibonacci Numbers," Fibonacci Quarterly, 1 (1963), No. 1, pp. 57-63.
2. V. C. Harris and Carolyn C. Styles, "A Generalization of Fibonacci Numbers," Fibonacci Quarterly, 2 (1964), pp. 277-289.
3. V. C. Harris and Carolyn C. Styles, "A Further Generalization of Fibonacci Numbers," Fibonacci Quarterly, 4(1966), pp. 241-248
4. V. E. Hoggatt, Jr., "A New Angle on Pascal's Triangle," to appear, Fibonacci Quarterly.
5. N. N. Vorobyov, The Fibonacci Numbers, D. C. Heath and Co., Boston, 1963.
6. Kathleen Weland, "Some Rabbit Production Results Involving Fibonacci Numbers," Fibonacci Quarterly, 5 (1967), pp. 195-200.
7. D. A. Lind and V. E. Hoggatt, Jr., "Fibonacci and Binomial Properties of Weighted Compositions," Journal of Combinatorial Theory 4, pp. 121–124 (1968).

* * * * *