PYTHAGOREAN TRIADS OF THE FORM X, X+1, Z
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T. W. FORGET and T. A. LARKIN
Lockheed Missiles & Space Company, Sunnyvale, California

The term Pythagorean Triples or Triads is applied to those integers which describe all right triangles with integral sides. The sub-class which is the subject of this paper, is restricted to those of sides \( x, x + 1, \sqrt{2x^2 + 2x+1} \).

It is obvious that the smallest such triangle has sides \( 3, 4, 5 \). The problem is to find a general method of sequential progress through the family of all such triangles. In the course of this development, and consequent to a solution of Pell's equation, it is shown that these triangles bear a curious relationship to a series which, with the exception of a single coefficient, is identical with the Fibonacci series.

It can be shown that in a right triangle \( x^2 + y^2 = z^2 \), primitive solutions are given by integers \( a, b \) such that \( x = a^2 - b^2 \), \( y = 2ab \) and \( z = a^2 + b^2 \) where \( a > b \), and \( (a, b) \) are relatively prime. This paper will be concerned with triangles in which \( y = x \pm 1 \), or \( x^2 + (x \pm 1)^2 = z^2 \), the primitive solutions of which also take this form.

A. If \( x \) is odd and

\[
x = a^2 - b^2 \quad \text{and} \quad x + 1 = 2ab,
\]

then

\[
-1 = a^2 - 2ab - b^2 \\
-1 = a^2 - 2ab - b^2 + b^2 - b^2 \\
-1 = a^2 - 2ab + b^2 - 2b^2 \\
-1 = (a - b)^2 - 2b^2
\]

B. If \( x \) is even and

\[
x = V2x \\
(2x+l) \\
\]
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PYTHAGOREAN TRIADS OF THE FORM \( x, x + 1, z \)

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\[ x = 2ab \text{ and } x + 1 = a^2 - b^2 \]

(Note: In A, \( x \) was odd and in B, \( x \) is even in order to account for all possibilities.)

then

\[ +1 = a^2 - 2ab - b^2 \]
\[ +1 = (a - b)^2 - 2b^2 \]

Let \( p = a - b \) and \( q = b \), then by A and B above

\[ \pm 1 = p^2 - 2q^2 . \]

Equation (1) is an example of Pell’s equation. By inspection, the smallest integral solution greater than zero of this equation is \( p = 1 \), \( q = 1 \).

Equation (1) can be factored into

\[ (p - q\sqrt{2}) (p + q\sqrt{2}) = \pm 1 \]

which, when raised to the \( n^{th} \) power, becomes

\[ (p - q\sqrt{2})^n (p + q\sqrt{2})^n = \pm 1 \]

Specifically

\[ (1 - \sqrt{2})^n (1 + \sqrt{2})^n = \pm 1 \]

since \( p = 1 \), \( q = 1 \) is a solution of equation (1).

Now let

\[ (2) \quad p_n + q_n\sqrt{2} = (1 + \sqrt{2})^n \]

then

\[ (3) \quad p_n - q_n\sqrt{2} = (1 - \sqrt{2})^n \]
Then, by solving these simultaneous equations,

\[ p_n = \frac{1}{2} \left[ (1 + \sqrt{2})^n + (1 - \sqrt{2})^n \right] \tag{4} \]

\[ q_n = \frac{1}{2\sqrt{2}} \left[ (1 + \sqrt{2})^n - (1 - \sqrt{2})^n \right] \tag{5} \]

Since \( p = 1, q = 1 \) is the smallest solution of equation (1), then the general solution is given by (2) or (3) above and, therefore, by (4) and (5). (This can be found in most texts on Number Theory.)

Adding equations (4), (5)

\[ p_n + q_n = \frac{1}{2\sqrt{2}} \left[ \sqrt{2}(1 + \sqrt{2})^n + \sqrt{2}(1 - \sqrt{2})^n + (1 + \sqrt{3})^n - (1 - \sqrt{3})^n \right] \]

\[ = \frac{1}{2\sqrt{2}} \left[ (\sqrt{2} + 1)(1 + \sqrt{2})^n - (1 - \sqrt{2})(1 - \sqrt{2})^n \right] \]

\[ = \frac{1}{2\sqrt{2}} \left[ (1 + \sqrt{2})^{n+1} - (1 - \sqrt{2})^{n+1} \right] \tag{6} \]

Since \( p_n = a - b \) and \( q_n = b \), then

\[ a = p_n + q_n \]

or

\[ a = q_{n+1} \]

and, of course,

\[ b = q_n \]
Equation (2) can be rewritten

\[ p_{n+1} + q_{n+1} \sqrt{2} = (1 + \sqrt{2})^{n+1} \]
\[ = (1 + \sqrt{2})^n (1 + \sqrt{2}) \]
\[ = (p_n + q_n \sqrt{2}) (1 + \sqrt{2}) \]
\[ = p_n + p_n \sqrt{2} + q_n \sqrt{2} + 2q_n \]
\[ = (p_n + 2q_n) + \sqrt{2}(p_n + q_n) \]

But

\[ p_n + q_n = q_{n+1} \]
\[ \therefore p_{n+1} = p_n + 2q_n \]

Rewriting equations (7), (6) and subtracting,

\[ (7.a) \quad p_{n-1} = p_{n-2} + 2q_{n-2} \]
\[ (6.a) \quad q_{n-1} = p_{n-2} + q_{n-2} \]
\[ (8) \quad p_{n-1} = q_{n-1} + q_{n-2} \]

Now rewriting equation (6)

\[ (6.b) \quad q_n = p_{n-1} + q_{n-1} \]

Substitute equation (8)

\[ (9) \quad q_n = q_{n-1} + q_{n-2} + q_{n-1} \]
\[ q_n = 2q_{n-1} + q_{n-2} \]

In both A and B above, the term 2ab was used, once for x and once for x + 1. If p and q satisfy \( p^2 - 2q^2 = -1 \), then \( x + 1 = 2ab \). If p and q satisfy \( p^2 - 2q^2 = +1 \), then \( x = 2ab \). Equations (2) and (3) state that the only way
for the negative portion of equation (1) to be satisfied is for \((1 - \sqrt{2})^n\) to be negative. If \((1 - \sqrt{2})^n\) is negative, then \(x + 1 = 2ab\); if \((1 - \sqrt{2})^n\) is posi-
tive, then \(x = 2ab\). Since \((1 - \sqrt{2})\) is a negative term \((\sqrt{2} > 1)\), \((1 - \sqrt{2})^n\)
is positive when \(n\) is even and negative when \(n\) is odd. Now the formula for
one side of the triangle becomes

\[
2q_n q_{n+1} = \begin{cases} 
ex & \text{for even values of } n \\
x + 1 & \text{for odd values of } n
\end{cases}
\]

We have now developed a recurrence relationship for the \(q\) terms in relation
to previous \(q\) terms (equation 9).

Except for the coefficient 2 of \(q_{n-1}\), this is the Fibonacci Series. Note
that in this same manner the expression \(p_n = 2p_{n-1} + p_{n-2}\) can also be proved.

Until now nothing has been formulated concerning the hypotenuse or \(z\)
term of the Pythagorean Triple. Since squaring and taking the root of very
large numbers is difficult, it would be advantageous to have a recursive for-
mula for the \(z\) terms. We propose to prove that

\[
z_n = q_{2n+1}
\]

is such a formula. Then any Pythagorean Triad of the form \(x, x + 1, z\) can
be found recursively by using equations (9), (10), and (11). Further, by use
of equation (6), any two consecutive \(q\) terms can be found and the sequence
proceeds from there. See Appendix A. Proof for equation (11) follows.

From \(A\) and \(B\) above, two conditions are possible, either \(x = a^2 - b^2\)
and \(x + 1 = 2ab\) or \(x = 2ab\) and \(x + 1 = a^2 - b^2\). In either case,

\[
x^2 + (x + 1)^2 = (a^2 - b^2)^2 + (2ab)^2
\]

As stated before,

\[
2ab = 2q_n q_{n+1}
\]

for the \(n^{th}\) triad. Also,
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\[ a^2 - b^2 = q_{n+1}^2 - q_n^2 \]

since

\[ a = q_{n+1} \text{ and } b = q_n \]

Then,

\[
\sqrt{x^2 + (x + 1)^2} = \left( q_{n+1}^2 - q_n^2 \right) + \left( 2q_n q_{n+1} \right) = q_{n+1}^2 - 2q_n^2 q_{n+1} + q_n^4 + 4q_n^2 q_{n+1}^2 = q_{n+1}^2 + 2q_n^2 q_{n+1} + q_n^4 = \left( q_{n+1}^2 + q_n^2 \right) \]

To prove equation (11) all that remains is to prove that

\[ q_{2n+1} = q_{n+1}^2 + q_n^2 \]

To do this we will prove by induction on \( k \) that

\[ q_{2n+1} = q_{k+2} q_{2n-k} + q_{k+1} q_{2n-(k+1)} \]

If \( k = 0 \)

\[ q_{2n+1} = 2q_2 + q_{2n-1} \]

\[ q_2 = 2q_{2n-1} + q_{2n-2} \]

\[ q_{2n+1} = 2 \left( 2q_{2n-1} + q_{2n-2} \right) + q_{2n-1} \]

If \( k = 1 \)
Notice now that \( q_{2n+1} \) is represented in terms of

\[
(q_3 = 5, \; q_{2n-1}), \; (q_2 = 2, \; \text{and} \; q_{2n-2}).
\]

Assume that the \( k^{th} \) relationship is of the form

\[
q_{2n+1} = q_{k+2}q_{2n-k} + q_{k+1}q_{2n-(k+1)}
\]

Certainly the first relationship is true as we have just shown. Assume the \( k^{th} \) relationship is true. Then,

\[
q_{2n+1} = q_{k+2}q_{2n-k} + q_{k+1}q_{2n-(k+1)}
\]

From equation (9) we know

\[
q_{2n-k} = 2q_{2n-k-1} + q_{2n-k-2}
\]

Then

\[
q_{2n+1} = q_{k+2} \left( q_{2n-k-1} + q_{2n-k-2} \right) + q_{k+1}q_{2n-k-1}
\]

\[
q_{2n+1} = 2q_{k+2}q_{2n-k-1} + q_{k+1}q_{2n-k-2} + q_{k+1}q_{2n-k-1}
\]

\[
q_{2n+1} = q_{2n-k-1} \left( 2q_{k+2} + q_{k+1} \right) + q_{k+2}q_{2n-k-2}
\]

Since

\[
2q_{k+2} + q_{k+1} = q_{k+3},
\]

\[
q_{2n+1} = q_{k+3}q_{2n-k-1} + q_{k+2}q_{2n-k-2}
\]

This is the \( (k + 1)^{st} \) relationship and this proves the general equation inductively. Specifically, when \( k = n - 1, \)
Then this completes the proof for equation (11).
| n  | $q_n$ | $2q_nq_{n+1}$ | $x_1$ = 3 | $x_2$ = 20 | $x_3$ = 119 | $x_4$ = 696 | $x_5$ = 4059 | $x_6$ = 23360 | $x_7$ = 137903 | $x_8$ = 803760 | $x_9$ = 4684659 | $x_{10}$ = 27304196 | $x_{11}$ = 159140519 | $x_{12}$ = 927538920 | $x_{13}$ = 5406093003 | $x_{14}$ = 3150919100 | $x_{15}$ = 183648021599 | $x_{16}$ = 1070387585472 | $x_{17}$ = 6238626641379 | $x_{18}$ = 36361380737780 | $x_{19}$ = 211929657785303 | $x_{20}$ = 1235216565974040 |
|----|------|--------------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1  | 1    | 4            | x_1 = 3  | x_2 = 20 | x_3 = 119| x_4 = 696| x_5 = 4059| x_6 = 23360| x_7 = 137903| x_8 = 803760| x_9 = 4684659| x_{10} = 27304196| x_{11} = 159140519| x_{12} = 927538920| x_{13} = 5406093003| x_{14} = 3150919100| x_{15} = 183648021599| x_{16} = 1070387585472| x_{17} = 6238626641379| x_{18} = 36361380737780| x_{19} = 211929657785303| x_{20} = 1235216565974040|
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**APPENDIX A (Continued)**

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### Appendix B

**Pythagorean Triads of the Form** $x$, $x + 1$, $z$

*Described by Recurrence Sequences*  

**June 1968**

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