ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets in the format used below. Solutions should be received within three months of the publication date.

B-142 Proposed by William D. Jackson, SUNY at Buffalo, Amherst, N. Y.

Define a sequence as follows: $A_1 = 2$, $A_2 = 3$, and $A_n = A_{n-1}A_{n-2}$ for $n > 2$. Find an expression for A_n.

B-143 Proposed by Raphael Finkelstein, Tempe, Arizona.

Show that the following determinant vanishes when a and d are natural numbers:

$$\begin{vmatrix} F_a & F_{a+d} & F_{a+2d} \\ F_{a+3d} & F_{a+4d} & F_{a+5d} \\ F_{a+6d} & F_{a+7d} & F_{a+8d} \end{vmatrix} .$$

What is the value of the determinant one obtains by replacing each Fibonacci number by the corresponding Lucas number?

B-144 Proposed by J. A. H. Hunter, Toronto, Canada

In this alphametic each distinct letter stands for a particular but different digit, all ten digits being represented here. It must be the Lucas series, but what is the value of the SERIES?

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      O N E
    T H R E E
    S T A R T
            L
  ___________
  S E R I E S
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B-145 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Given an unlimited supply of each of two distinct types of objects, let f(n) be the number of permutations of n of these objects such that no three consecutive objects are alike. Show that $f(n) = 2F_{n+1}$, where F_n is the n^{th} Fibonacci number.

B-146 Proposed by Walter W. Horner, Pittsburgh, Pennsylvania.

Show that $\pi = \operatorname{Arctan}(1/F_{2n}) + \operatorname{Arctan} F_{2n+1} + \operatorname{Arctan} F_{2n+2}$.

B-147 Proposed by Edgar Karst, University of Arizona, Tucson, Arizona, in honor of the 66th birthday of Hansraj Gupta on Oct. 9, 1968.

Let

$$S = (1/3 + 1/5) + (1/5 + 1/7) + \cdots + (1/32717 + 1/32719)$$

be the sum of the sum of the reciprocals of all twin primes below 2^{15}.. Indicate which of the following inequalities is true:

(a) $S \leq \pi^2/6$, (b) $\pi^2/6 < S < \sqrt{e}$ (c) $\sqrt{e} < S$.

SOLUTIONS

NOTE: The name of A. C. Shannon was inadvertently omitted from the list of solvers of B-109.

LINEAR COMBINATION OF GEOMETRIC SERIES

B-124 Proposed by J.H.Butchart, Northern Arizona University, Flagstaff, Arizona.

Show that

$$\sum_{i=0}^{\infty} (a_i / 2^i) = 4 ,$$

where

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 2, \cdots$$

are the Fibonacci numbers.

Solution by R. L. Mercer, University of New Mexico, Albuquerque, N. Mex.

Convergence of the series follows from

$$\lim_{n \to \infty} (a_{n+1} / a_n) = (1 + \sqrt{5})/2$$

and the ratio test. Let T be the value of the series. Then

$$T = \sum_{i=0}^{\infty} (a_{i+2} - a_{i+1})/2^i = 4 \sum_{i=0}^{\infty} a_{i+2} / 2^{i+2} - 2 \sum_{i=0}^{\infty} a_{i+1} / 2^{i+1}$$

and

$$T = 4(T - a_0 - a_1/2) - 2(T - a_0) .$$

Solving, we find

$$T = 2(a_0 + a_1) = 2a_2 = 4 .$$

Also solved by Dewey C. Duncan, Bruce W. King, J. D. E. Konhauser, F. D. Parker, C. B. A. Peck, A. C. Shannon (Australia), John Wessner, David Zeitlin, and the proposer.

EDITORIAL NOTE:

Since

$$f(x) = \frac{1}{1 - x - x^2} = \sum_{n=0}^{\infty} F_{n+1} x^n = \sum_{i=0}^{\infty} a_i x^i$$

Substituting

$$x = \tfrac{1}{2} < \frac{\sqrt{5} - 1}{2} = \left(\frac{1 + \sqrt{5}}{2}\right)^{-1}$$

yields

$$f(\tfrac{1}{2}) = \sum_{i=0}^{\infty} a_i / 2^i = \frac{1}{1 - \frac{1}{2} - \frac{1}{4}} = 4 ,$$

while $f(-1/2) = 4/5$.

V. E. H.

A NON-INTEGRAL SUM

B-125 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Is

$$\sum_{k=3}^{n} \frac{1}{F_k}$$

ever an integer? Explain.

Solution by Dewey C. Duncan, Los Angeles, California.

The summation

$$\sum_{k=3}^{n} \frac{1}{F_k}$$

is never an integer, since

(1) For $n = 3, 4, 5$, the summation yields $1/2$, $5/6$, $31/30$, respectively.

(2) For $n > 5$, the summation yields a sum that is greater than 1 and less than 1.5, since

$$\frac{F_{2k-1}}{F_{2k}} > \frac{F_{2k+1}}{F_{2k+2}}, \qquad \frac{F_{2k}}{F_{2k+1}} < \frac{F_{2k+2}}{F_{2k+3}},$$

and

$$\lim_{n=\infty} \frac{F_n}{F_{n+1}} = \frac{\sqrt{5}-1}{2}$$

From

$$F_{2k-1}F_{2k+1} - F_{2k}^2 = (-1)^{2k}$$

one implies that for all $k \geq 1$,

$$\frac{F_{2k}}{F_{2k+2}} < \frac{F_{2k}}{F_{2k+1}} < \frac{F_{2k-1}}{F_{2k}} < \frac{F_{2k-1}}{F_{2k}}$$

Therefore, since

$$\frac{F_3}{F_4} = \frac{2}{3},$$

we conclude that, for $k > 3$,

$$\frac{F_k}{F_{k+1}} < \frac{2}{3}.$$

Consequently,

$$\sum_{k=3}^{\infty} \frac{1}{F_k} < \frac{1}{2}\left[1 + (2/3) + (2/3)^2 + (2/3)^3 + \cdots\right] ,$$

whence,

$$\sum_{k=3}^{\infty} \frac{1}{F_k} < 3/2 \quad \text{Q.E.D.}$$

Also solved by R. L. Mercer, C. B. A. Peck, and the proposer.

GOOD ADVICE

B-126 Proposed by J. A. H. Hunter, Toronto, Canada.

Each distinct letter in this alphametic stands, of course, for a particular and different digit. The advice is sound, for our FQ is truly <u>prime</u>. What do you make of it all?

$$\begin{array}{r} \text{READ} \\ \text{FQ} \\ \text{READ} \\ \underline{\text{FQ}} \\ \text{DEAR} \end{array}$$

Solution by Charles W. Trigg, San Diego, California.

From the units' column R is even. Since $2R + 1 = D$, then $(R,D) = (2,5)$ or $(4,9)$.

If $(R,D) = (4,9)$, then (since FQ is prime) $Q = 3$ and $F = 1, 2, 5, 7$, or 8. Furthermore, $2F + A + 2$ is a multiple of ten. Thus $(F,A) = (1,6)$, $(5,8)$ or $(8,2)$. But each of these pairs leads to a value of E which duplicates another digit.

If $(R,D) = (2,5)$ then (since FQ is prime) $Q = 1$, and $F = 3, 4, 6$ or 7. Now $2F + A + 1$ is a multiple of ten, so $(F, A) = (6, 7)$ is the sole solution. Whereupon $E = 10 - 2$ or 8. The unique reconstruction of the addition is

$$\begin{array}{r} 2875 \\ 61 \\ 2875 \\ \underline{61} \\ 5872 \end{array}$$

Additional solution by David Zeitlin, Minneapolis, Minnesota.

$$\begin{array}{r} 0841 \\ 79 \\ 0841 \\ \underline{79} \\ 1840 \end{array}$$

Also solved by H. D. Allen (Canada), A. Gommel, R. L. Mercer, John W. Milsom, C. B. A. Peck, and the proposer.

CONGRUENCES

B-127 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tennessee.

Show that

$$2^n L_n \equiv 2 \pmod 5,$$
$$2^n F_n \equiv 2n \pmod 5.$$

Solution by John Wessner, Melbourne, Florida.

We proceed by induction. Both results are true for $n = 1, 2$. If we assume that the first for $n = k$ and $n = k + 1$, then we have

$$2^k L_k \equiv 2 \pmod 5, \quad 2^{k+1} L_{k+1} \equiv 2 \pmod 5.$$

Combining these,

$$2^{k+2} L_{k+2} = 2(2^{k+1} L_{k+1} + 2 \cdot 2^k L_k) \equiv 2(2 + 2 \cdot 2) \equiv 2 \pmod 5.$$

Similarly, in the second case we assume

$$2^k F_k \equiv 2k \pmod 5, \quad 2^{k+2} F_{k+2} \equiv 2(k+1) \pmod 5 .$$

Combining these gives

$$\begin{aligned} 2^{k+2} F_{k+2} &= 2(2^{k+1} F_{k+1} + 2 \cdot 2^k F_k) \\ &\equiv 2 \; 2(k+1) + 2 \cdot 2k \quad \equiv 12k + 4 \\ &\equiv 2(k+2) \pmod 5 . \end{aligned}$$

Also solved by Herta T. Freitag, R. L. Mercer, C. B. A. Peck, A. C. Shannon (Australia), Paul Smith (Canada), David Zeitlin, and the proposer.

GENERALIZED SEQUENCES

B-128 Proposed by M. N. S. Swamy, Nova Scotia Technical College, Halifax, Canada.

Let f_n be the generalized Fibonacci sequence with $f_1 = a$, $f_2 = b$, and $f_{n+1} = f_n + f_{n-1}$. Let g_n be the associated generalized Lucas sequence defined by $g_n = f_{n-1} + f_{n+1}$. Also let $S_n = f_1 + f_2 + \cdots + f_n$. It is true that $S_4 = g_4$ and $S_8 = 3g_6$. Generalize these formulas.

Solution by C. B. A. Peck, Ordnance Research Laboratory, State College, Pennsylvania.

By induction,

$$S_n = f_{n+2} - f_2 , \qquad f_n = F_{n-1} f_2 + F_{n-2} f_1 ,$$

and

$$g_n = L_{n-1} f_2 + L_{n-2} f_1 .$$

Thus

$$S_{4n} = f_{4n-2} - f_2 = (F_{4n-1} - 1) f_2 = F_{4n} f_1$$

and

$$F_{2n}g_{2n+2} = F_{2n}(L_{2n+1}f_2 + L_{2n}f_1) .$$

These are equal, since

$$F_{4n} = F_{2n}L_{2n}$$

and

$$F_{4n-1} - 1 = F_{2n}L_{2n+1} .$$

Thus we have

$$S_{4n} = F_{2n}g_{2n+2} .$$

P. S. $S_n = f_{n+2} - f_2$ occurs in B-20, FQ, Vol. 2, pp. 76-77.

Also solved by Bruce W. King, A. C. Shannon (Australia), David Zeitlin, and the proposer.

MODIFIED GOLDEN RATIO

B-129 Proposed by Thomas P. Dence, Bowling Green State University, Bowling Green, Ohio.

For a given positive integer, k, find

$$\lim_{n\to\infty} (F_{n+k} / L_n) .$$

Solution by Bruce W. King, Burnt Hills — Balston Lake H.S., Burnt Hills, N. Y.

Let $a = (1 + \sqrt{5})/2$ and $b = (1 - \sqrt{5})/2$. Then $|b/a| < 1$ and it follows that $(b/a)^n \to 0$ as $n \to \infty$. Hence

$$F_{n+k} / L_n = (a^{k+k} - b^{n+k})/\sqrt{5}(a^n + b^n) = (a^k/\sqrt{5})[1 - (b/a)^{n+k}]/[1 + (b/a)^n]$$

approaches $a^k/\sqrt{5}$ as n goes to infinity.

Also solved by R. L. Mercer, C. B. A. Peck, A. C. Shannon (Australia), Paul Smith (Canada), John Wessner, and the proposer.

MULTINOMIAL COEFFICIENTS

B-130 Proposed by Douglas Lind, University of Virginia, Charlottesville, Va.

Let coefficients $c_j(n)$ be defined by

$$(1 + x + x^2)^n = c_0(n) + c_1(n)x + c_2(n)x^2 + \cdots + c_{2n}(n)x^{2n}$$

and show that

$$\sum_{j=0}^{2n} c_j(n)^2 = c_{2n}(2n) .$$

Generalize to

$$(1 + x + x^2 + \cdots + x^k)^n ,$$

Solution by David Zeitlin, Minneapolis, Minnesota.

Let

$$\begin{aligned} Q(x) = Q_{k,n}(x) &= (1 + x + x^2 + \cdots + x^k)^n \\ &= q_0(n) + q_1(n)x + q_2(n)x^2 + \cdots + q_{kn}(n)x^{kn} . \end{aligned}$$

Since

$$x^{kn}Q(1/x) = Q(x) ,$$

we have

$$q_j(n) = q_{kn-j}(n)$$

for $j = 0, 1, \cdots, kn$. Equating coefficients of x^{kn} in

$$(1 + x + \cdots + x^k)^{2n} = \left[(1 + x + \cdots + x^k)^n\right]^2 ,$$

we obtain

$$q_{kn}(2n) = \sum_{r=0}^{kn} q_r(n)q_{kn-r}(n) = \sum_{r=0}^{kn} [q_r(n)]^2 .$$

Also solved by R. L. Mercer, R. W. Mercer, A. C. Shannon (Australia), and the proposer.

A FIBONACCI-LUCAS IDENTITY

B-131 Proposed by Charles R. Wall, University of Tennessee, Knoxville, Tennessee

Prove that for m odd

$$\frac{L_{n-m} + L_{n+m}}{F_{n-m} + F_{n+m}} = \frac{5F_n}{L_n}$$

and for m even

$$\frac{F_{n-m} + F_{n+m}}{L_{n-m} + L_{n+m}} = \frac{F_n}{L_n} .$$

Solution by John Wessner, Melbourne, Florida.

The following properties of the Fibonacci and Lucas numbers can easily be proved by the use of the Binet formula: (1) For odd values of m,

$$L_{n-m} + L_{n+m} = 5F_nF_m ,$$

$$F_{n-m} + F_{n+m} = L_nF_m .$$

(2) for even values of m,

$$L_{n-m} + L_{n+m} = L_nL_m ,$$

$$F_{n-m} + F_{n+m} = F_nL_m .$$

[Continued on p. 304.]

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