PARTITIONS OF N INTO DISTINCT FIBONACCI NUMBERS

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INTRODUCTION

Suppose $\{a_n\}$ is a sequence of natural numbers such that $a_{n+2} = a_{n+1} + a_n$, $n = 1, 2, \cdots$, and let $A(n)$ be the number of sets of numbers $\{i_1, i_2, \cdots\}$ such that $n = a_{i_1} + a_{i_2} + \cdots$. When $a_n = F_n$, F_{n+1} or L_n (where as usual F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively) we write $A(n) = R(n)$, $T(n)$, or $S(n)$, respectively. Among other things we proved the following theorems in an earlier paper on this subject [4].

<u>Theorem 1.</u> If $a_n \le K = a_n + k \quad a_{n+1} - a_2$, $n = 3, 4, \cdots$, then

(a) $$A(K) = A(k) + A(a_{n-1} - k - a_2) ,$$

and

(b) $$A(K) = A(a_{n+1} - k - a_2) .$$

Also, if $a_2 \ge 2$ and $1 \le k \le a_2 - 1$, then

(c) $$A(a_{n-1} + k - a_2) = A(a_n - k) = A(a_{n+1} + k - a_2) , \quad n = 4, 5, \cdots .$$

<u>Theorem 2:</u>

(a) $T(N) = 1$ if, and only if, $N = F_{n+1} - 1$, $n = 0, 1, \cdots$.

(b) $T(N) = 2$ if, and only if, $N = F_{n+3} + F_n - 1$ or $F_{n+4} - F_n - 1$, $n = 1, 2, \cdots$.

(c) $T(N) = 3$, if and only if, $N = F_{n+5} + F_n - 1$, $F_{n+5} + F_{n+1} - 1$, $F_{n+6} - F_n - 1$, or $F_{n+6} - F_{n+1} - 1$, $n = 1, 2, \cdots$.

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(d) $T(F_{n+k+2} + 2F_{n+2} - 1) = k, \quad n = 1, 2, \cdots,$ and $k = 4, 5, \cdots$.

For several values of k Hoggatt found solution sets of $T(x) = k$; in each case this solution set could be described as a finite set of sequences having the form $b_n - 1$ where $b_{n+2} = b_{n+1} + b_n$. Thus he was led to conjecture: If $\{b_n\}$ is a sequence of natural numbers such that $b_{n+2} = b_{n+1} + b_n$, then

$$T(b_n - 1) = T(b_{n+1} - 1) = k$$

for all sufficiently large n. Our main purpose in this note is to give proof of Hoggatt's conjecture.

A REPRESENTATION THEOREM

Suppose $\cdots, F_{-1}, F_0, F_1, \cdots$ is the extended sequence of Fibonacci members; that is, $F_0 = 0$, $F_1 = 1$, and $F_{n+1} - F_n - F_{n-1} = 0$, $-\infty \le n \le \infty$. Thus, we have

$$F_{-n} = (-1)^{n+1} F_n, \quad n = 1, 2, \cdots .$$

The following representation theorem should be compared with Zeckendorf's theorem (see for example Brown [1], [2], or Daykin [3]); in particular, is there a sequence essentially different from $\{F_n\}$ which satisfies the conditions of Theorem 3?

Theorem 3. For every pair of non-negative integers A and B there exists a unique set of integers $\{k_1, \cdots, k_i\}$ such that $|k_r - k_s| \ge 2$ whenever $r \ne s$, and

$$A = F_{k_1} + \cdots + F_{k_i} \text{ and } B = F_{k_1+1} + \cdots + F_{k_i+1} .$$

Proof. If a set of integers $\{m_1, \cdots, m_i\}$ has $|m_r - m_s| \ge 2$ whenever $r \ne s$, $F_{m_1} + \cdots + F_{m_i}$ is called a minimal sum. There is a finite algorithm $\mathfrak{A}$ for converting $F_m + F_{m_1} + \cdots + F_{m_i}$ into a minimal sum if $F_{m_1} + \cdots + F_{m_i}$ is a minimal sum $\mathfrak{A}$: First, if $m = m_j$ for some j we can convert $F_{m_1} + \cdots + 2F_{m_j} + \cdots + F_{m_i}$ into a sum involving F's with distinct subscripts since there is a maximal t such that $2F_m + F_{m-2} + \cdots + F_{m-2t}$ is a part of this sum, and this can be replaced with

$$F_{m+1} + F_{m-1} + \cdots + F_{m-2t+1} + F_{m-2t-2} \; .$$

Second, if $F_m + F_{m_1} + \cdots + F_{m_i}$ is a sum involving F's with distinct subscripts, a minimal sum can be obtained in a finite number of steps by successively replacing $F_v + F_{v-1}$, v maximal, with F_{v+1}. Note that if $\mathfrak{A}$ is applied to $F_{n+m_1} + \cdots + F_{n+m_1}$ and $F_{n+n_1} + \cdots + F_{n} + n_j$ is the result when $n = 0$, then the same statement holds for $n = 1, 2, \cdots$.

Consider the sequence $\{b_n\}$ defined by

$$b_0 = A, \quad b_1 = B, \quad b_{n+2} = b_{n+1} + b_n, \quad n = 0, 1, \cdots ,$$

then it follows that

$$b_n = F_{n-1}A + F_nB, \quad n = 0, 1, \cdots .$$

Using the algorithm $\mathfrak{A}$ we are going to show by induction on $A + B$ that for every pair of non-negative integers A, B there exists a unique set of integers $\{k_1, \cdots, k_i\}$ such that $|k_r - k_s| \geq 2$ when $r \neq s$, and

$$AF_{n-1} + BF_n = F_{n+k_1} + \cdots + F_{n+k_i}, \quad n = 0, 1, \cdots . \tag{1}$$

If $A + B = 1$, then

$$AF_{n-1} + BF_n$$

is F_{n-1} or F_n, $n = 0, 1, \cdots$. Suppose the statement is true for every pair of non-negative integers A, B with $A + B \leq n$ $(n \geq 1)$. Then if $A + B = n$, there exists a unique set of integers $\{k_1, \cdots, k_i\}$ with $|k_r - k_s| \geq 2$ when $r \neq s$, and

$$AF_{n-1} + BF_n = F_{n+k_1} + \cdots + F_{n+k_i} \, .$$

Now we can apply $\mathfrak{A}$ to

$$(A + 1)F_{n-1} + BF_n = F_{n-1} + F_{n+k_1} + \cdots + F_{n+k_i}$$

or

$$AF_{n-1} + (B+1)F_n = F_n + F_{n+k_1} + \cdots + F_{n+k_i}$$

to find that there is at least one set of integers which satisfies (1) for every pair of non-negative integers A,B with $A + B = n + 1$. But suppose $AF_{n-1} + BF_n$ can be expressed as a minimal sum in two ways for $n = 0, 1, \cdots$, say

$$AF_{n-1} + BF_n = F_{n+r_1} + \cdots + F_{n+r_i} = F_{n+s_1} + \cdots + F_{n+s_j} .$$

Thus, for every

$$n \geq \max\{r_1, \cdots, r_i, s_1, \cdots, s_j\}$$

the number $AF_{n-1} + BF_n$ has two representations as a sum of non-consecutive Fibonacci numbers (with positive subscripts); this contradicts Zeckendorf's theorem which says that such representations are unique for every natural number.

Corollary: If $\{b_n\}$ is a sequence of natural numbers such that

$$b_{n+2} = b_{n+1} + b_n, \quad n = 0, 1, \cdots ,$$

then there exists a unique set of integers $\{k_1, \cdots, k_i\}$ with $|k_r - k_s| \geq 2$ when $r \neq s$, such that

$$b_n = F_{n+k_1} + \cdots + F_{n+k_i}, \quad n = 0, 1, \cdots . \tag{2}$$

Proof. Put $b_0 = A$, $b_1 = B$ in Theorem 3, then (2) can be proved by induction on n.

HOGGATT'S CONJECTURE

Theorem 4. Suppose $\{b_n\}$ is a sequence of natural numbers such that $b_{n+2} = b_{n+1} + b_n$, then there exists an N such that

$$T(b_n - 1) = T(b_{n+1} - 1), \quad n \geq N; \tag{3}$$

in fact, if

$$(4) \qquad b_n = F_{n+k_1} + \cdots + F_{n+k_i}, \quad k_j \geq k_{j+1} + 2, \quad j = 1, \cdots, i-1.$$

then $N = 2 - k_i$. If $k_i > 2$, the extended sequence found by substituting $n = -1, \cdots, 2 - k_i$ in (4) satisfies (3) for $n \geq 2 - k_i$.

Proof. The Corollary to Theorem 3 guarantees that b_n has the (unique) representation given in (4), so we can assume b_n has this form. If $i = 1$, Theorem 2(a) asserts $T(F_n - 1) = 1$ for $n = 1, 2, \cdots$, so

$$T(F_{n+k_1} - 1) = T(F_{n+k_1+1} - 1)$$

for $n \geq 2 - k_1$ (in fact for $n \geq 1 - k_1$). Now assume $i > 1$. We have

$$F_{n+k_1} \leq b_n - 1 \leq F_{n+k_1+1} - F_3,$$

for $n \geq 3 - k_1 \geq 2 - k_i$, so Theorem 1(a) can be used to write

$$(5) \qquad T(b_n - 1) = T(b_n - F_{n+k_1} - 1) + T(F_{n+k_1+1} - b_n + 1 - F_3).$$

Suppose $1 \leq j \leq i$ is the smallest member such that $k_j > k_{j+1} + 2$, then

$$(6) \qquad F_{n+k_1+1} - b_n + 1 - F_3 = \begin{cases} F_{n+k_i-1} - 1, & \text{if } j = i, \\ F_{n+k_j-2} + F_{n+k_{j+1}} + \cdots + F_{n+k_i} - 1, & \text{if } j < 1. \end{cases}$$

Now (5) and (6) indicate that Theorem 4 can be proved by means of a double induction on i and $k_1 - k_2 = k \geq 2$; thus, for $i, k \geq 2$ we define proposition $P(i,k)$: If $\{b_n\}$ is a sequence of natural numbers with

$$b_n = F_{n+k_1} + \cdots + F_{n+k_i},$$

such that $k_1 \geq k_2 + 2, \cdots, k_{i-1} \geq k_i + 2$, and $k_1 - k_2 = k$, then

$$T(b_n - 1) = T(b_{n+1} - 1)$$

for all $n \geq 2 - k_i$.

To prove $P(2, 2)$ is true, suppose

$$b_n = F_{n+k_1} + F_{n+k_2}$$

with $k_2 = k_1 - 2$; then using (5) and (6) we have

$$T(F_{n+k_1} + F_{n+k_2} - 1) = T(F_{n+k_2} - 1) + T(F_{n+k_2-1} - 1) , \tag{7}$$

but

$$T(F_{n+k_2} - 1) = T(F_{n+k_2-1} - 1) = 1$$

for all $n \geq 2 - k_2$.

Suppose $P(2, k)$ is true for all $k < K$ $(K > 2)$, and suppose

$$b_n = F_{n+k_1}, \ F_{n+k_2}$$

with $k_1 - k_2 = K$, then using (5) and (6) we have

$$T(F_{n+k_1} + F_{n+k_2} - 1) = T(F_{n+k_2} - 1) + T(F_{n+k_1-2} + F_{n+k_2} - 1) . \tag{8}$$

If

$$K = 3, \quad T(F_{n+k_2} - 1) = T(F_{n+k_1-2} + F_{n+k_2} - 1) = T(F_{n+k_1+1} - 1) = 1 ,$$

for all $n \geq 2 - k_2$. If $K > 3$,

$$T(F_{n+k_1-2} + F_{n+k_2} - 1) = T(F_{n+k_1+1} + F_{n+k_2+1} - 1) \quad \text{for all } n \geq 2 - k_2 ,$$

so $P(2,k)$ is true; thus, $P(2,k)$ is true for all $k \geq 2$.

Now we suppose $P(i,k)$ is true for all $i < I$ $(I > 2)$ and all $k > 2$; there is no difficulty in showing that $P(I,2)$ is true and that $P(I,K-1)$ implies $P(I,K)$ for $K > 2$, by using (5) and (6) just as before. This completes the proof.

Corollary:

$$T(F_{n+k} + F_n - 1) = \left[\frac{k+2}{2}\right], \quad k,n = 2,3,\cdots .$$

Proof: Combining (7) and (8) and related results we have

$$T(F_{n+k} + F_n - 1) = \begin{cases} 2, & \text{if } k = 2,3, \\ 1 + T(F_{n+k-2} + F_n - 1), & \text{if } k = 4,5,\cdots . \end{cases} \tag{9}$$

The proof follows by induction on k in (9).

Theorem 5. Suppose $\{b_n\}$ is a sequence of natural numbers such that

$$b_{n+2} = b_{n+1} + b_n ,$$

then $T(b_n)$, $T(b_{n+2})$, $\cdots$, and $R(b_n)$, $R(b_{n+2}),\cdots$ form arithmetic progressions for all sufficiently large n.

Proof. The proof that $T(b_n)$, $T(b_{n+2}),\cdots$ forms an arithmetic progression follows the proof of Theorem 4, except that we use the fact that a term-by-term sum of two arithmetic progressions is also an arithmetic progression. Theorem 4 and this last result imply $R(b_n)$, $R(b_{n+2}),\cdots$ forms an arithmetic progression because $R(N) = T(N) + T(N-1)$, so $R(b_n) + T(b_n - 1)$.

SOLVING $T(x) = j$

In the last section we showed that $T(x) = T(y)$ for every pair $x, y \in S(k_1,\cdots,k_i) = \{F_{n+k_1} + \cdots + F_{n+k_i} - 1; n = 2-k_1, 3-k_1,\cdots\}$, where

$$k_1 \geq k_2 + 2, \cdots, k_{i-1} \geq k_i + 2 ;$$

since

$$S(k_1, \cdots, k_i) = S(k_1 + k, \cdots, k_i + k),$$

we will assume $k_i = 0$. The next theorem asserts that every solution x of $T(x) = j$ is contained in one of a finite collection of sets $S(k_1, \cdots, k_i)$ for appropriate sets of numbers $\{k_1, \cdots, k_i\}$.

<u>Theorem 6.</u> (a) Every non-negative integer is contained in exactly one of the sets $S(k_1, \cdots, k_i)$, where $\{k_1, \cdots, k_i\}$ ranges over all sets of integers such that

$$k_1 \geq k_2 + 2, \cdots, k_{i-1} \geq k_i + 2, \quad k_i = 0 .$$

(b) If x, y $S(k_1, \cdots, k_i)$, then

$$T(x) = T(y) \leq \left[\frac{k_1 + 2}{2}\right] .$$

(c) There exists a finite, non-empty collection of sets $S(r_1, \cdots, r_m)$, $S(s_1, \cdots, s_m), \cdots$ such that $T(x) = j$ if, and only if, $x \in S(r_1, \cdots, r_m) \cup S(s_1, s_n) \cup \cdots$.

<u>Proof.</u> (a) This is a reformulation of Zeckendorf's Theorem. (b) The result is true when $i = 1$ or 2 by Theorem 2(a) and the Corollary to Theorem 4, respectively. Now (5) and (6) can be used to prove (b) by induction; the main point of the proof is indicated by the following inequality:

$$(10) \qquad \begin{aligned} T(F_{n+k_1} + \cdots + F_{n+k_i} - 1) &= T(F_{n+k_2} + \cdots + F_{n+k_1} - 1) \\ &\quad + T(F_{n+k_j-2} + \cdots + F_{n+k_i} - 1) \\ &\geq \begin{cases} 1 + \left[\frac{k_1}{2}\right] = \left[\frac{k_1 + 2}{2}\right], & \text{if } j = 1 , \\ \left[\frac{k_1}{2}\right] + \left[\frac{k_1 - 2j + 2}{2}\right] \geq \left[\frac{k_1 + 2}{2}\right], & \text{if } j > 1. \end{cases} \end{aligned}$$

(c) Every number is in exactly one of the sets $S(k_1, \cdots, k_i)$ by (a) of this Theorem; but $x \in S_j = \{x : T(x) = j\}$ and $x \in S(k_1, \cdots, k_i)$ implies $S(k_1, \cdots, k_i)$ is contained in S_j since $T(x) = T(y) = j$ for every $y \in S(k_1, \cdots, k_i)$ by Theorem 4. There are only finitely many sets $\{k_1, \cdots, k_i\}$ such that

$$k_1 \geq k_2 + 2, \cdots, k_{i-1} \geq k_i + 2, \quad k_i = 0,$$

and

$$\left[\frac{k_1 + 2}{2}\right] \leq j,$$

so S_j is a finite union of sets $S(r_1, \cdots, r_m)$, $S(s_1, \cdots, s_n), \cdots$. The corollary of Theorem 4 implies S_j is non-empty for $j = 1, 2, \cdots$; a different collection of solutions of $T(x) = j$ was given in [4].

Let $t(k_1, \cdots, k_i) = T(x)$, where $x \in S(k_1, \cdots, k_i)$; then if $i = 1$, we have $t(0) = 1$ which is Theorem 2(a). For $i > 1$, if j is the smallest number such that $k_j > k_{j+1} + 2$, then (5) and (6) may be formulated as

$$(11) \qquad t(k_1, \cdots, k_i) = \begin{cases} t(k_2, \cdots, k_i) + 1, & \text{if } j = i \\ t(k_2 + \cdots + k_i) + t(k_j - 1, k_{j+2}, \cdots, k_i) & \\ \quad \text{if } j < i, \ k_j = k_{j+1} + 3, & \\ t(k_2 + \cdots + k_i) + t(k_j - 2, k_{j+1}, \cdots, k_i) & \\ \quad \text{if } j < i, \ k_j \geq k_{j+1} + 4 . & \end{cases}$$

Using Theorem 6(b) and (11) we can find all solutions of $T(x) = j$ with a finite amount of checking. This checking would be made easier if we had a non-iterative method for computing $t(k_1, \cdots, k_i)$, but so far we have not been able to find a closed formula for $t(k_1, \cdots, k_i)$.

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MORE ABOUT THE "GOLDEN RATIO" IN THE WORLD OF ATOMS

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In an earlier article (The Fibonacci Quarterly, Issue 4, 1963) the author reported some fundamental asymmetries that appear in the world of atoms.

It has been stated in this article that the numerical values of all these asymmetries approximately are equal to the "golden ratio" ("g. r.").

Two of these asymmetries were found:

1. In the structure of atomic nuclei of protons and neutrons, and
2. In the distribution of nucleons in fission-fragments of the heaviest nuclei appearing in some nuclear reactions.

Recent theoretical studies suggest that an element containing 114 protons and 184 neutrons may be comparitively stable and therefore this hypothetical substance could be produced possibly in some nuclear reactions [1].

One possible reaction involves bombarding element 92 (uranium) with ions (atoms stripped of one or more electrons) of the same element 92, which should yield a hypothetical compound nucleus ${}_{184}[x]^{476}$ that could break up asymmetrically and produce a nucleus with 114 protons:

$${}_{92}U^{238} + {}_{92}U^{238} \rightarrow {}_{184}[x]^{476} \rightarrow {}_{114}[y]^{298} + {}_{70}Yb^{166} + 12n;$$

12 neutrons (n) would be left over from the reaction [2].

Remark: Both hypothetical (with no names) products of this reaction are designated with the symbols [x] and [y] respectively.

It turns out that the ratio of 114 protons and 184 (298 - 144 = 184) neutrons of the hypothetical element 114 is equal to 0.6195 and differs from the "g. r."-value (if we limit the "g. r."-value to four decimals behind the point) by 0.0015 only.

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