EVEN PERFECT NUMBERS AND SEVEN

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Over the years number theory has given both professional and amateur mathematicians many hours of frustration and enjoyment. The study of perfect numbers is an area of the theory of numbers which dates back to antiquity.

A positive integer n is a perfect number if and only if the sum of its positive integer divisors is $2n$. For example, 28 is a perfect number since the positive integer divisors of 28 are 1, 2, 4, 7, 14, and 28 and

$$1 + 2 + 4 + 7 + 14 + 28 = 56 = 2 \times 28 .$$

The first few perfect numbers are 6; 28; 496; 8,128; 33,550,336; and 8,589,056. Notice that each of these perfect numbers is even. Although no odd perfect number has ever been found, mathematicians have been unable to prove that none exists. It is also unknown whether or not the number of perfect numbers is infinite.

Euclid showed that if n is a positive integer of the form $2^{p-1}(2^p - 1)$ where $2^p - 1$ is a prime then n is a perfect number. Later Euler established that every even perfect number is of the Euclid type. A necessary condition that $2^p - 1$ be a prime is that p be a prime. Thus all even perfect numbers have the form $2^{p-1}(2^p - 1)$ where p is a prime number.

If $p = 3$ then $2^{p-1}(2^p - 1) = 28$ which is a multiple of 7. Since 3 is the only multiple of three which is a prime number, all other prime numbers are of the form $3j + 1$ or $3j + 2$. A careful investigation of the even perfect numbers different from 28 given above yields the following table.

p	$2^{p-1}(2^p - 1)$	p	$2^{p-1}(2^p - 1)$
2	$6 = 7 \cdot 0 + 6$	7	$8128 = 7 \cdot 1161 + 1$
5	$496 = 7 \cdot 70 + 6$	13	$33550336 = 7 \cdot 4792905 + 1$
17	$8589869056 = 7 \cdot 1227124150 + 6$		

This leads us to conjecture that if $n = 2^{p-1}(2^p - 1)$ is an even perfect number different from 28 then n is of the form $7k + 1$ or $7k + 6$ according as p is of the form $3j + 1$ or $3j + 2$. Before attempting to prove this conjecture, we shall establish some preliminary results.

Lemma 1. For each positive integer w, $2^{3w} = 7t + 1$ for some positive integer t.

Proof. $2^3 = 8 = 7 \cdot 1 + 1$. Assume that $2^{3w} = 7r + 1$. Then

$$2^{3(x+1)} = 2^{3x+3} = 2^{3x} \cdot 2^3 = (7r + 1)8 = 7(8r + 1)$$

and the lemma follows by the principle of mathematical induction.

Lemma 2. For each nonnegative integer z, $2^{3z+1} = 7s + 3$ for some nonnegative integer s.

Proof. If $z = 0$ then $2^{3z+1} = 2 = 7 \cdot 0 + 2$. Assume that $2^{3y+1} = 7m + 2$. Then

$$2^{3(y+1)+1} = 2^{(3y+1)+1} = 2^{3y+1} \cdot 2^3 = (7m + 2)8 = 7(8m + 2) + 2$$

and the lemma follows by the principle of mathematical induction.

Theorem. If $n = 2^{p-1}(2^p - 1)$ is an even perfect number different from 28 then n is of the form $7k + 1$ or $7k + 6$ according as p is of the form $3j + 1$ or $3j + 2$.

Proof. $n \neq 28$ implies that $p \neq 3$. Since p is a prime number and $p \neq 3$, p is either of the form $3j + 1$ or $3j + 2$.

Case 1. $p = 3j + 1$. Then $p - 1 = 3j$ and $2^{p-1} = 2^{3j} = 7t + 1$ by Lemma 1. Hence $2^p = 2 \cdot 2^{p-1} = 14t + 2$, from which it follows that $2^p - 1 = 14t + 1$. Thus $n = 2^{p-1}(2^p - 1) = (7t + 1)(14t + 1) = 7(14t^2 + 3t) + 1$.

Case 2. $p = 3j + 2$. Then $p - 1 = 3j + 1$ and $2^{p-1} = 2^{3j+1} = 7s + 2$ by Lemma 2. Hence $2^p = 2 \cdot 2^{p-1} = 14s + 4$, from which it follows that $2^p - 1 = 14s + 3$. Thus $n = 2^{p-1}(2^p - 1) = (7s + 2)(14s + 3) = 7(14s + 7s) + 6$.

Let n be an even perfect number. It can be shown that if $n \neq 6$ then n yields the remainder 1 when divided by 9; if $n \neq 6$ and $n \neq 496$ then n ends with 16, 28, 36, 56, or 76 when n is written in base 10 notation; and if $n \neq 6$ then n has the remainder 1, 2, 3, or 8 when divided by 13.

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