THE GENERALIZED FIBONACCI OPERATOR

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I. INTRODUCTION

Some years ago Angus E. Taylor and the author were looking for examples of operators for which spectra could be determined and classified. In the course of this search we chanced upon a bounded linear operator F on the sequence space ℓ_1, defined by the infinite matrix (f_{ij}),

$$f_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i = 1, \; j > 1 \\ 0 & \text{otherwise} \end{cases}$$

This operator has the interesting property that the norms of its consecutive powers are consecutive Fibonacci numbers, which, as is well known, are defined recursively by

$$f_0 = 0, \quad f_1 = 1 \quad \text{and} \quad f_n = f_{n-1} + f_{n-2}, \qquad n \geq 2 .$$

The infinite matrix representations of the n^{th} power of this operator have column vectors such that the first $n+1$ terms of these vectors are, in inverted order, truncated Fibonacci sequences. The spectrum consists of the unit disc together with the point

$$\frac{1 + \sqrt{5}}{2} ,$$

the positive zero of the polynomial $P(\lambda) = \lambda^2 - \lambda - 1$, sometimes called the "golden mean" which is well known to be the limit, as n becomes infinite of the positive n^{th} root of the n^{th} term of the Fibonacci sequence. We appropriately enough dubbed this operator the "Fibonacci Operator."

In this paper we define an operator-valued function F of a nonnegative real variable, such that for every nonnegative value of x there is associated with the number x a bounded linear operator $F(x)$ on the sequence space ℓ_1. In addition, there corresponds to each nonnegative value of x:

(1) a sequence $\{f_k(x)\}$

(2) a polynomial $P_x(\lambda)$

(3) an infinite matrix representation $(f_{ij}(x))$ for $F(x)$.

For the case $x = 1$, $F(1)$, $\{f_k(1)\}$, $P_1(\lambda)$ and $(f_{ij}(1))$ are the Fibonacci operator, the Fibonacci sequence, the associated polynomial, and matrix representation, respectively. For all other values of x, $0 \leq x < \infty$, $F(x)$ and the entities referred to in (1), (2), and (3) above have interrelationships similar to those possessed by their counterparts in the case $x = 1$.

II. PRELIMINARY DEFINITIONS AND NOTATION

The operators we shall consider will be bounded linear operators mapping the sequence space ℓ_1 into itself. The space ℓ_1 consists of the set of all absolutely convergent sequences of complex numbers $\xi = \{\xi_i\}$ under the norm defined by

$$\|\xi\| = \sum_{i=1}^{\infty} |\xi_i| .$$

It can be shown (see for example [1]) that every member, A, of the algebra $[\ell_1]$ of bounded linear operators which map ℓ_1 into itself has a matrix representation (a_{ij}), such that the uniform norm of A is given by

$$\|A\| = \sup_j \sum_{i=1}^{\infty} |a_{ij}| .$$

If A is in $[\ell_1]$, then the resolvent set of A, $\rho(A)$, consists of the set of all complex λ for which the operator $(\lambda I - A)^{-1}$, where I is the identity operator, exists as a bounded operator, and the range of $\lambda I - T$ is dense in ℓ_1.

The spectrum of A, $\sigma(A)$, consists of the set of all complex numbers which do not belong to $\rho(A)$. The spectral radius of A, $|\sigma(A)|$, is the radius of the smallest circle, with center at the origin, which contains $\sigma(A)$. We shall have occasion to make use of the following facts: (see 2)

(2.1) $$|\sigma(A)| = \lim_{n\to\infty} \|A^n\|^{1/n}$$

(2.2) If $|\lambda| \geq |\sigma(A)|$ we can represent $(\lambda I - A)^{-1}$ by its Neumann expansion,

$$(\lambda I - A)^{-1} = \frac{I}{\lambda} + \sum_{n=1}^{\infty} \frac{A^n}{\lambda^{n+1}} \quad .$$

The function F which we wish to consider has for its domain the set of all non-negative real numbers and its range is contained in $[\ell_1]$. If we identify the values of F(x) with their matrix representations under the standard basis, it will be convenient to define F(x) as the sum of two matrices L and C(x).

The infinite matrix $L = (\ell_{ij})$ is defined by

$$\ell_{ij} = \begin{cases} 1 & \text{if } i - j = 1 \\ 0 & \text{otherwise} \end{cases} .$$

When L is used as a left multiplier on a matrix A, we might call it a "lowering matrix." Its effect on A can be crudely described as follows: Each row of A is lowered one step, and the empty first row is replaced by zeros.

The infinite matrix $C(x) = (c_{ij}(x))$ is defined by

$$c_{ij}(x) = \begin{cases} 0 & \text{if } j < [x] + 1 \text{ or } i > 1 \\ j - x & \text{if } j = [x] + 1 \text{ and } i = 1 \\ 1 & \text{if } j > [x] + 1 \text{ and } i = 1 \end{cases} ,$$

where $[x]$ denotes the greatest integer not greater than x. (Note that all entries of C(x) below the first row are zero.) This matrix could be described as "partial column summer." As a left multiplier of a matrix $A = (a_{ij})$, it produces the following effect. In each column of A the elements below the

$[x+1]^{st}$ row are summed, to this is added $(1 - x + [x])$ times the entry in the $[x+1]^{st}$ row and the total is entered as the first row entry of the corresponding column of LA. All other entries in this column of LA are 0.

We are now ready to state our main theorem.

III. PRINCIPAL THEOREM

Theorem 1. Let $F(x)$ be the member of $[\ell_1]$ defined by the infinite matrix $L + C(x)$, $0 \le x < \infty$. With $F(x)$ there are associated

(1) a sequence $\{f_k(x)\}$, defined by

$$f_k(x) = \begin{cases} 0 \text{ if } k = 0 \\ 1 \text{ if } 0 < k \le [x+1] \\ f_{k-1}(x) + ([x+1] - x)f_{k-[x+1]}(x) + (x - [x])f_{k-[x+2]}(x) \\ \qquad \text{if } k > [x] + 1 \end{cases}$$

and

(2) a polynomial $P_x(\lambda)$,

$$P_x(\lambda) = \left\{ \lambda^{[x+1]} - ([x+1] - x)\right\}(\lambda - 1) - 1$$

such that the following relationships hold.

(a)
$$\left\| F^n(x) \right\| = f_{n+[x+2]}(x) - ([x+1] - x)f_{n+1}(x) = \sum_{k=0}^{n} f_k^{(x)} + 1$$

(b)
$$\sigma(F) = \left\{ \lambda;\ P(\lambda) = 0 \text{ or } |\lambda| \le 1 \right\}$$

(c)
$$\lim_{n\to\infty} \left\{ f_n(x) - ([x+1] - x)f_{n-[x+1]}(x) \right\}^{1/n} = \left|\sigma(F(x))\right| = \sup_{P(\lambda)=0} |\lambda|$$

(d)
$$f_j(x) = f^{(k)}_{(j+1-k)n}(x), \quad j = 1, \cdots, k, \qquad n > [x+1]$$

where

$$F^k(x) = (f_{ij}^{(k)}(x)) .$$

Statement (d) merely says that the first k entries in any column after the $[x+1]^{st}$ of the matrix $F^k(x)$ are the truncated sequence $\{f_k\}_1^k$ in reverse order.

Before proceeding with the proof, we note that in case x is an integer, the sequence $\{f_k(x)\}$ is a sequence of integers similar to the Fibonacci sequence; indeed $\{f_k(1)\}$ is the Fibonacci sequence and $\{f_k(0)\}$ starting with $f_1(0)$ is the geometric progression with first term equal to 1 and common ratio 2. In general, where x is an integer, the sequence $\{f_k(x)\}$ has the following properties:

(i) $$f_0(x) = 0, \quad f_1(x) = f_2(x) = \cdots = f_{x+1}(x) = 1$$

$$f_n(x) = f_{n-1}(x) + f_{n-(1+x)}(x) \text{ if } n > x+1$$

(ii) $$f_{n+x+1}(x) = \sum_{k=0}^{n} f_k(x) + 1$$

(iii) $$\lim_{n\to\infty} \{f_n(x)\}^{1/n} = \sup_{P(\lambda)=0} |\lambda| = |\sigma(F)| ,$$

where

$$P(\lambda) = (\lambda^{x+1} - 1)(\lambda - 1) - 1$$
$$= \lambda^{x+2} - \lambda^{x+1} - \lambda$$

(iv) $$f_{n+x+1}(x) = \|F^n(x)\| .$$

We now turn to the proof of our theorem.

We shall let the matrix representation of $F^n(x)$ be denoted by $(f_{ij}^{(n)}(x))$. However, to simplify the notation in the discussion that follows, we will omit the argument x and represent $F^n(x)$ and $f_{ij}^{(n)}(x)$ merely by F^n and $f_{ij}^{(n)}$. We shall also let

$$\epsilon = 1 - (x - [x])$$

and

$$\ell = \epsilon + x = 1 + [x] .$$

With this notation the matrix representation of F(x) has the appearance:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & \epsilon & 1 & 1 & 1 & \cdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots \\ & \cdots & & & & & \cdots & & & & \end{bmatrix},$$

where ϵ appears in the first row of the ℓ^{th} column.

Since $F^n = (L + C)F^{n-1}$, we see from the description of the effects produced by L and C as left operators that the k^{th} row of F^n is the first row of F^{n-k+1} for $1 \le k \le n$, k an integer. That is

$$f_{kj}^{(n)} = f_{(k-1)j}^{(n-1)} = \cdots = f_{1j}^{(n-k+1)}; \quad 1 \le k \le n. \tag{3.1}$$

We also see that

$$f_{kj}^{(n)} = \begin{cases} 1 & \text{if } k = n + j \\ 0 & \text{if } k > n \text{ and } k \neq n + j . \end{cases} \tag{3.2}$$

With the understanding that if $n \le 0$ then $f_{1j}^{(n)} = 0$ and $f_{1n}^{(1)} = f_{1n} = 0$ we can state the following lemma.

Lemma 1.

(a) $$f_{1m}^{(n)} = \epsilon f_{1m}^{(n-\ell)} + \sum_{j=1}^{n-\ell-1} f_{1m}^{(j)} + f_{1(m+n-1)}$$

if m and n are positive integers.

(b) $$f_{1m}^{(n)} = f_{1m}^{(n-1)} + \epsilon f_{1m}^{(n-\ell)} + (1-\epsilon) f_{1m}^{(n-\ell-1)} + (f_{1(m+n-1)} - f_{1(m+n-2)})$$

if m and n are positive integers and $n \geq 2$.

Proof. Part (a) follows easily from the fact that

$$f_{1m}^{(n)} = \sum_{j=1}^{\infty} f_{1j} f_{jm}^{(n-1)}$$

and formulas 3.1 and 3.2. Part (b) is obtained by computing $f_{1m}^{(n-1)}$ from part (a) and subtracting the result from the expression for $f_{1m}^{(n)}$ given in (a).

Lemma 2. If n is an integer and $n \geq 2$ then

$$f_{1m}^{(n)} = g(m) f_{11}^{(n-1)} + f_{1(m+1)}^{(n-1)} ,$$

where

$$g(m) = \begin{cases} 0 & \text{if } m < \ell \\ \epsilon & \text{if } m = \ell \\ 1 & \text{if } m > \ell . \end{cases}$$

Proof. The result follows easily from the fact that

$$f_{1m}^{(n)} = \sum_{j=1}^{\infty} f_{1j}^{(n-1)} \cdot f_{jm} .$$

Lemma 3. If m, n and k are positive integers and $m > k$, then $f_{1m}^{(n)} \geq f_{1k}^{(n)}$. If in addition $k > \ell$, then $f_{1k}^{(n)} = f_{1m}^{(n)}$.

Proof. This result follows from an inductive argument. That the conclusions of the lemma hold for $n = 1$ is evident. From the induction hypotheses that they hold for $n = j$, it quickly follows from Lemma 2 that they hold for $n = j + 1$.

Since from Lemma 3, $f_{1k}^{(n)} = f_{1m}^{(n)}$ for all k and m such that $k > \ell$ and $m > \ell$, we make the following definition.

Definition.

$$f_0 = 0, \quad f_n = f_{1(\ell+1)}^{(n)} \quad \text{if } n \text{ is a positive integer.}$$

From the definitions of ℓ and ϵ and Lemmas 1 and 3, it follows that $\{f_n(x)\}$ is the sequence defined in Part 1 of the conclusion of Theorem 1.

Lemma 4. The norm of F^n is given by

$$F^n = \sum_{k=0}^{n} f_k + 1$$

Proof. Since $0 \le \epsilon \le 1$ and all the entries of the first row of the matrix (f_{ij}) are nonnegative, it follows from part (b) of Lemma 1 that all the elements of the first row of the matrix $(f_{ij}^{(n)})$ are nonnegative. From equation 3.1 we see that the j^{th} component of the m^{th} column vector of $(f_{ij}^{(n)})$ is given by

$$f_{jm}^{(n)} = f_{1m}^{(n-j+1)}, \quad 1 \le j \le n .$$

From this equation and equation 3.2 it follows since all the components are nonnegative that the ℓ_1 norm of the m^{th} column vector of $(f_{ij}^{(n)})$ is given by

$$\sum_{j=1}^{n} f_{1m}^{(j)} + 1 .$$

From Lemma 3 we see that the ℓ_1 norm of the $(\ell + 1)^{st}$ column vector of $(f_{ij}^{(n)})$ is greater than or equal to the ℓ_1 norm of any other column vector of that matrix. The definition of $\|F^n\|$ and that of the sequence $\{f_k\}$ now imply that

$$\|F^n\| = \sum_{k=0}^{n} f_k + 1 .$$

This completes the proof of Lemma 4.

It is a simple matter to use the result part (b) of Lemma 1 to conclude that

$$\sum_{k=0}^{n} f_k + 1 = f_{n+\ell+1} - \epsilon f_{n+1} .$$

This result together with Lemma 4 gives part (a) of part 2 of the conclusion of Theorem 2.

Lemma 5. The formal inverse matrix (g_{ij}) of the matrix representation of $\lambda I - F$ is defined by

$$b_{1j} = \frac{\lambda^{\ell+1}}{P(\lambda)} \begin{cases} \dfrac{\lambda\epsilon + (1-\epsilon)}{\lambda^{\ell-j+3}} & \text{if } j \le \ell \\ \dfrac{1}{\lambda^2} & \text{if } j > \ell \end{cases}$$

where

$$P(\lambda) = \lambda^{\ell+1} - \lambda^{\ell} - \epsilon\lambda - (1-\epsilon) ,$$

and

$$g_{ij} = \begin{cases} \dfrac{1}{\lambda^{i-j+1}} + \dfrac{1}{\lambda^{i-1}} b_{1j} & \text{if } i \ge j \\ \dfrac{1}{\lambda^{i-1}} b_{1j} & \text{if } i < j . \end{cases}$$

Proof. The Neumann expansion for $(\lambda I - F)^{-1}$ converges provided $|\lambda| \geq |\sigma(F)|$. Since

$$|\sigma(F)| = \lim_{n \to \infty} \|F^n\|^{1/n} \leq \|F\| = 2,$$

it is clear that the Neumann expansion for $(\lambda I - F)^{-1}$ converges provided $|\lambda| \geq 2$. We, however, are only using the Neumann expansion as a device to obtain the formal matrix inverse of the matrix representation of $(\lambda I - F)$. If we let the matrix for $(\lambda I - F)^{-1}$ be denoted by (g_{ij}), then since

$$(\lambda I - F)^{-1} = \frac{I}{\lambda} + \sum_{n=1}^{\infty} \frac{F^n}{\lambda^{n+1}}$$

it follows that

$$g_{ij} = \frac{\delta_{ij}}{\lambda} + \sum_{n=1}^{\infty} \frac{f_{ij}^{(n)}}{\lambda^{n+1}} .$$

But from 3.1 and 3.2 we see that:

$$f_{ij}^{(n)} = \begin{cases} f_{1j}^{(n-i+1)} & \text{if } 1 \quad i \leq n, \\ \delta_{i(j+n)} & \text{if } i > n . \end{cases}$$

Thus

$$\text{(3.3)} \qquad g_{ij} = \begin{cases} \displaystyle\sum_{n=i}^{\infty} \frac{f_{1j}^{(n-i+1)}}{\lambda^{n+1}} & \text{if } i < j , \\ \displaystyle\frac{1}{\lambda^{i-j+1}} + \sum_{n=i}^{\infty} \frac{f_{1j}^{(n-i+1)}}{\lambda^{n+1}} & \text{if } i \geq j . \end{cases}$$

If we now consider the matrix (g_{ij}) as the sum of two matrices (a_{ij}) and (b_{ij}) where

$$a_{ij} = \begin{cases} 0 & \text{if } i < j , \\ \dfrac{1}{\lambda^{i-j+1}} & \text{if } i \geq j . \end{cases} \tag{3.4}$$

we see that

$$b_{ij} = \sum_{n=i}^{\infty} \frac{f_{1j}^{(n-i+1)}}{\lambda^{n+1}} .$$

If $i > 1$ we see that

$$b_{ij} = \sum_{n=i}^{\infty} \frac{f_{1j}^{(n-i+1)}}{\lambda^{n+1}} = \sum_{k=1}^{\infty} \frac{f_{1j}^{(k)}}{\lambda^{k+i}} \tag{3.5}$$

$$= \frac{1}{\lambda^{i-1}} \sum_{k=1}^{\infty} \frac{f_{1j}^{(k)}}{\lambda^{k+1}} = \frac{1}{\lambda^{i-1}} b_{1j} .$$

By using part b of Lemma 1, we can solve for values b_{1j} as follows:

$$b_{1j} = \sum_{k=1}^{\infty} \frac{f_{ij}^{(k)}}{\lambda^{k+1}} = \frac{f_{1j}}{\lambda^2} + \sum_{k=2}^{\infty} \frac{f_{1j}^{(k)}}{\lambda^{k+1}}$$

$$= \frac{f_{1j}}{\lambda^2} + \sum_{k=2}^{\infty} \frac{\left\{ f_{1j}^{(k-1)} + \epsilon f_{1j}^{k-\ell} + (1-\epsilon) f_{1j}^{(k-\ell-1)} + f_{1(j+k-1)} - f_{1(j+k-2)} \right\}}{\lambda^{k+1}}$$

or

$$b_{1j} = \frac{f_{1j}}{\lambda^2} + \frac{1}{\lambda} b_{1j} + \frac{\epsilon}{\lambda^{\ell}} b_{1j} + \frac{(1-\epsilon)}{\lambda^{\ell+1}} b_{1j}$$

$$+ \begin{cases} \dfrac{\epsilon}{\lambda^{\ell-j+2}} + \dfrac{1-\epsilon}{\lambda^{\ell-j+3}} & \text{if } j \le \ell \\[2ex] \dfrac{1}{\lambda^2} & \text{if } j > \ell, \end{cases}$$

and therefore

$$(3.6) \qquad b_{1j} = \frac{\lambda^{\ell+1}}{\lambda^{\ell+1} - \lambda - \epsilon\lambda - (1-\epsilon)} \cdot \begin{cases} \dfrac{\lambda\epsilon + (1-\epsilon)}{\lambda^{\ell-j+3}} & \text{if } j \le \ell \\[2ex] \dfrac{1}{\lambda^2} & \text{if } j > \ell \end{cases} .$$

Remembering that $g_{ij} = a_{ij} + b_{ij}$ the conclusion of the lemma follows from equations 3.3, 3.4, 3.5, and 3.6.

From Lemma 5 it is easy to see that the matrix (g_{ij}) can be schematically presented as the linear combination of two matrices as follows:

$$(g_{ij}) = \begin{bmatrix} \frac{1}{\lambda} & 0 & 0 & 0 & \cdots & & \\ \frac{1}{\lambda^2} & \frac{1}{\lambda} & 0 & 0 & \cdots & & \\ \frac{1}{\lambda^3} & \frac{1}{\lambda^2} & \frac{1}{\lambda} & 0 & \cdots & & \\ & \cdots & & & \cdots & & \\ \frac{1}{\lambda^k} & \frac{1}{\lambda^{k-1}} & \cdots & & \frac{1}{\lambda} & 0 & 0 \\ & \cdots & & & \cdots & & \end{bmatrix} + \frac{\lambda^{\ell-1}}{P(\lambda)} \begin{bmatrix} h^{(1)}(\epsilon) & h^{(2)}(\epsilon) & \cdots & h^{(\ell)}(\epsilon) & 1 & 1 & \cdots \\ \frac{h^{(1)}(\epsilon)}{\lambda} & \frac{h^{(2)}(\epsilon)}{\lambda} & \cdots & \frac{h^{(\ell)}(\epsilon)}{\lambda} & \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots \\ \frac{h^{(1)}(\epsilon)}{\lambda^2} & \frac{h^{(2)}(\epsilon)}{\lambda^2} & \cdots & \frac{h^{(\ell)}(\epsilon)}{\lambda^2} & \frac{1}{\lambda^2} & \frac{1}{\lambda^2} & \cdots \\ \frac{h^{(1)}(\epsilon)}{\lambda^3} & \frac{h^{(2)}(\epsilon)}{\lambda^3} & \cdots & \frac{h^{(\ell)}(\epsilon)}{\lambda^3} & \frac{1}{\lambda^3} & \frac{1}{\lambda^3} & \cdots \\ \cdots & & & \cdots & & & \end{bmatrix},$$

where

$$h^{(j)}(\epsilon) = \frac{\lambda\epsilon + (1-\epsilon)}{\lambda^{\ell+1-j}}$$

is a factor of each element in each of the first ℓ columns of the second matrix. The first of the above matrices is the matrix representation of

$$I + \sum_{k=1}^{\infty} L^k$$

and the value of its norm is

$$\frac{1}{|\lambda| - 1} = \sum_{k=1}^{\infty} \frac{1}{|\lambda|^k},$$

if $|\lambda| > 1$. The value of the norm of the second matrix is

$$\max \begin{cases} |h^{(j)}(\epsilon)| \dfrac{|\lambda|}{|\lambda| - 1} = |h^{(j)}(\epsilon)| + \displaystyle\sum_{k=1}^{\infty} \dfrac{|h^{(j)}(\;)|}{|\lambda|^k}, \quad j = 1, \cdots, \ell \\ \dfrac{|\lambda|}{|\lambda| - 1} = 1 + \displaystyle\sum_{k=1}^{\infty} \dfrac{1}{|\lambda|^k} \end{cases}$$

provided $|\lambda| > 1$.

From these facts we can infer that $(\lambda I - F)^{-1}$ is defined and a bounded operator on ℓ_1 into ℓ_1 provided $|\lambda| > 1$ and λ is not a zero of $P(\lambda)$, and that $(\lambda I - F)^{-1}$ is either not defined or is unbounded if λ is a zero of $P(\lambda)$ or $|\lambda| \leq 1$. We thus conclude that the resolvent set of F,

$$\rho(F) = \{\lambda \mid |\lambda| > 1 \text{ and } P(\lambda) \neq 0 \},$$

and therefore the spectrum of F,

$$\sigma(F) = \{\lambda \mid |\lambda| \leq 1 \text{ or } P(\lambda) = 0\}.$$

This proves part 2.b of our theorem since if we recall that $\ell = [x+1]$ and $\epsilon = ([x+1] - x)$ we see that the polynomial $P(\lambda)$ is precisely the polynomial $P_x(\lambda)$ defined in the theorem by:

$$P_x(\lambda) = \{\lambda^{[x+1]} - ([x+1] - x)\}(\lambda - 1) - 1 .$$

Lemma 6. For any given value of x, $0 \le x \le \infty$, $P_x(\lambda)$ has precisely one real zero, r_x, with modulus greater than 1 and $1 \le r_x \le 2$.

Proof. As a function of the real variable ξ

$$P_x(\xi) = (\xi^{\ell} - \epsilon)(\xi - 1) - 1 = \xi^{\ell+1} - \xi^{\ell} - \epsilon\xi - (1 - \epsilon)$$

and

$$P'_x(\xi) = (\xi^{\ell} - \epsilon) + (\xi - 1)\ell\xi^{\ell-1} = (\ell + 1)\xi^{\ell} - \ell\xi^{\ell-1} - \epsilon .$$

It is a simple matter to verify that $P'_x(\xi) > 0$ if $\xi > 1$ and $P_x(1) = -1$. From this we infer that $P_x(\xi)$ has precisely one zero greater than 1 and since $P_x(2) > 0$, that this zero lies strictly between 1 and 2 if $x \neq 0$. If $\xi < -1$ and ℓ is odd then $P'_x(\xi) < 0$ and $P_x(-1) > 0$. If $\xi < -1$ and ℓ is even, then $P'_x(\xi) > 0$ and $P_x(-1) > 0$. From these facts it follows that $P_x(\xi)$ has no negative zeros with modulus greater than 1. This completes the proof of the lemma.

Lemma 7. If r_x is the positive real zero of $P_x(\lambda)$, $1 < r_x \le 2$, and if μ is any other zero of $P_x(\lambda)$, then $|\mu| \le r_x$.

Proof. The proof is by contradiction. If we assume $P_x(\mu) = 0$ and $|\mu| > r_x > 1$, then $|\mu|^{\ell} > r_x^{\ell}$ and therefore $|\mu|^{\ell} - \epsilon > r_x^{\ell} - \epsilon > 0$ since $0 \le \epsilon \le 1$. From this last result the following chain of inequalities follows:

$$\frac{1}{|\mu|^{\ell} - \epsilon} < \frac{1}{r_x^{\ell} - \epsilon} ,$$

hence

$$1 + \frac{1}{|\mu|^{\ell} - \epsilon} < 1 + \frac{1}{r_x^{\ell} - \epsilon} = r_x$$

since

$$r_x^{\ell+1} - r_x^{\ell} - \epsilon r_x - (1 - \epsilon) = 0 ,$$

and therefore

$$1 + \frac{1}{|\mu^{\ell} - \epsilon|} < r_x ,$$

or

$$|\mu| < r_x ,$$

since

$$|\mu| = \left| 1 + \frac{1}{\mu^{\ell} - \epsilon} \right| \leq 1 + \frac{1}{|\mu^{\ell} - \epsilon|} .$$

But $|\mu| < r_x$ is a contradiction of our assumption that $\mu \geq r_x$.

From Lemmas 6 and 7 and the definition of spectral radius, we immediately deduce the second equality in part 2.c of the conclusion of Theorem 1. That is,

$$|\sigma(F(x))| = \sup_{P_x(\lambda)=0} |\lambda|$$

The first equality of part 2.c of the conclusion of Theorem 1 is an immediate consequence of part 2.a of Theorem 1 and the fact, 2.1, that

$$|\sigma(F(x))| = \lim_{n \to \infty} \|F^n(x)\|^{1/n} .$$

We have now completed the proof of Theorem 1.

IV. A PROPERTY OF $|\sigma(F(x))|$

We conclude this paper with the following theorem.

Theorem 2. The spectral radius of $F(x)$ is a strictly decreasing continuous function of x, $x \geq 0$, and

(a) $$\lim_{x \to \infty} |\sigma(F(x))| = 1$$

(b) $$\lim_{x \to 0} |\sigma(F(x))| = 2 .$$

Proof. From Theorem 1 we know that

$$|\sigma(F(x))| = r_x$$

where r_x is the only real root of $P_x(\xi)$, $|\xi| > 1$, and $1 < r_x \leq 2$. Let us assume that n is a positive integer and

$$n - 1 \leq x < y < n .$$

It now follows that $r_x > r_y$. The proof is by contradiction.

Assume $r_x \leq r_y$. Then

$$P_x(r_x) = (r_x^n - \epsilon_x)(r_x - 1) - 1 = 0$$

and

$$P_y(r_y) = (r_y^n - \epsilon_y)(r_y - 1) - 1 = 0$$

where

$$\epsilon_x = [x + 1] - x .$$

From these equations and the assumption that $r_x \leq r_y$, it follows that

$$r_x = 1 + \frac{1}{r_x^n - \epsilon_x} \leq 1 + \frac{1}{r_y^n - \epsilon_y} = r_y ,$$

or

$$r_y^n - r_x^n \le \epsilon_x - \epsilon_y = x - y < 0$$

and therefore $r_y < r_x$ which is a contradiction to our assumption that $r_x \le r_y$.

Since we have shown that r_x is strictly decreasing as x increases and is therefore strictly increasing as x decreases for

$$n - 1 < x < n ,$$

we see that if

$$n - 1 < y < n,$$

then the following limits exist:

$$\lim_{x \to y+} r_x = \alpha \text{ and } \lim_{x \to y-} r_x = \beta .$$

Therefore, since

$$\lim_{x \to y} \epsilon_x = \epsilon_y ,$$

$$\lim_{x \to y+} P_x(r_x) = P_y(\alpha) = 0 \lim_{x \to y-} P_x(r_x) = P_y(\beta) .$$

But since

$$P_y(\xi), \quad |\xi| > 1 ,$$

has only one real root, namely r_y, it follows that $r_y = \alpha = \beta$ and therefore

$$\lim_{x \to y} r_x = r_y$$

or r_x is a continuous function of x on

$$n - 1 < x < n .$$

It is not difficult to see that

$$\lim_{x \to n} r_x = r_n$$

where n is any positive integer. First it is clear that as

$$x \to n^+, \quad P_x(r_x) = (r_x^{n+1} - \epsilon_x)(r_x - 1) - 1 = 0$$

and

$$\epsilon_x = (n + 1) - x ,$$

provided

$$x < n + 1 ,$$

hence

$$\lim_{x \to n+} P_x(r_x) = (\gamma^{n+1} - 1)(\gamma - 1) - 1 = \gamma^{n+2} - \gamma^{n+1} - \gamma = 0 = P_n(\gamma)$$

where

$$\lim_{x \to n+} r_x = \gamma .$$

Similarly as $x \to n^-$,

$$P_x(r_x) = (r_x^n - \epsilon_x)(r_x - 1) - 1 = 0$$

and

$\epsilon_x - n - x$, provided $x > n - 1$, hence

$$\lim_{x \to n^-} P_x(r_x) = (\delta^n - 0)(\delta - 1) - 1 = \delta^{n+1} - \delta^n - 1 = 0 = \frac{P_n(\delta)}{\delta},$$

where

$$\lim_{x \to n^-} r_x = \delta .$$

Since both γ and δ must lie between 1 and 2 and $P_n(\xi)$, $|\xi| > 1$, has precisely one real root we infer that $\gamma = \delta$ or r_x is continuous at $x = n$ for n an arbitrary positive integer.

It now follows that r_x is a continuous function of x for all $x > 0$ and r_x is a strictly decreasing function of x.

Finally we shall show that

$$\lim_{x \to \infty} r_x = 1 .$$

For assume

$$\lim_{x \to \infty} r_x = r$$

where $r > 1$. In this case

$$\lim_{x \to \infty} \left\{ r_x^{[x+1]} - \epsilon_x \right\} = \lim_{x \to \infty} \frac{1}{r_x - 1} = \frac{1}{r - 1}$$

since for all $x > 0$

$$r_x^{[x+1]} - \epsilon_x = \frac{1}{r_x - 1} .$$

But it is clear that

$$r_x^{[x+1]} - \epsilon_x$$

becomes arbitrarily large as x approaches infinity and therefore cannot have

$$\frac{1}{r-1}$$

as a limit. This contradicts our assumption that $r > 1$.

That

$$\lim_{x \to 0} r_x = 2$$

follows immediately from the fact that

$$P_0(\lambda) = \lambda^2 - 2\lambda .$$

REFERENCES

1. L. W. Cohen and Nelson Dunford, "Transformations on Sequence Spaces," Duke Math. J., 3 (1937), 689-701.

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A CURIOUS PROPERTY OF A SECOND FRACTION

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In the April, 1968 Fibonacci Quarterly (p. 156), J. Wlodarski discussed some properties of the fraction 878/323 which approximates e. Consider the approximation of π correct to six decimal places given by $355/113 = 3.141592^+$. The sum of the digits of the numerator is 13, and of the denominator, 5. $13/5 = 1 + 8/5$, or one added to the best approximation to the "Golden Ratio" using two one-digit numbers. Also,

$$\frac{355}{113} = \frac{300 + 55}{100 + 13} ,$$

where 55 and 13 are Fibonacci numbers.

Taking 355/226 as an approximation of $\pi/2$ leads to the observation that

$$\frac{355}{226} = \frac{377 - 22}{233 - 7}$$

where 377/233 approximates the golden ratio and 22/7 approximates π, and 377 and 233 are Fibonacci numbers.