SYMBOLIC SUBSTITUTIONS INTO FIBONACCI POLYNOMIALS

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1. INTRODUCTION

Symbolic equations give a compact way of representing certain identities. For example, if F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively, then two familiar identities may be written

$$(1 + F)^n = F^{2n}, \quad F^k \equiv F_k ,$$

$$(1 + L)^n = L^{2n}, \quad L^k \equiv L_k ,$$

where the additional qualifiers $F^k \equiv F_k$, $L^k \equiv L_k$ indicate that we drop exponents to subscripts after expanding. Further material on symbolic relations is given in [6, Chapter 15] and [7]. Here we make a similar "symbolic substitution" of certain sequences into the Fibonacci polynomials. We then find the auxiliary polynomials of the recurrence relations which the resulting sequences obey. Finally, we extend these results to the substitution of any recurrent sequence into any sequence of polynomials obeying a recurrence relation with polynomial coefficients.

2. SYMBOLIC SUBSTITUTION OF FIBONACCI NUMBERS INTO FIBONACCI POLYNOMIALS

The Fibonacci numbers F_n are defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n,$$

and the Lucas numbers L_n by

$$L_1 = 1, \; L_2 = 3, \; L_{n+2} = L_{n+1} + L_n .$$

Define the Fibonacci polynomials $f_n(x)$ by

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_{n+2}(x) = xf_{n+1}(x) + f_n(x) .$$

Consider the sequence $\{a_n\}$ given by

$$a_n = f_n(F), \quad F^k \equiv F_k ,$$

that is, a_n is the symbolic substitution of the Fibonacci numbers into the n^{th} Fibonacci polynomial. The first few terms are

$$a_1 = 0, \quad a_2 = 1, \quad a_3 = 1, \quad a_4 = 4, \quad a_5 = 6 .$$

We give four distinct methods of finding the recurrence relation obeyed by the a_n.

The first method applies a technique used by Gould [3]. Write the Fibonacci polynomials as in Figure 1. Our approach to find a_n is to multiply

$$\begin{array}{lllll}
1 & & & & \\
x & & & & \\
x^2 & +\ 1 & & & \\
x^3 & +\ 2x & & & \\
x^4 & +\ 3x^2 & +\ 1 & & \\
x^5 & +\ 4x^3 & +\ 3x & & \\
x^6 & +\ 5x^4 & +\ 6x^2 & +\ 1 & \\
\cdot & & & & \\
\cdot & & & & \\
\cdot & & & &
\end{array}$$

Figure 1

the coefficient of x^r by F_r and sum the coefficients in the n^{th} row. Now it is known [10] that

$$(1) \qquad f_n(x) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-j-1}{j} x^{n-2j-1} ,$$

where $[x]$ represents the greatest integer contained in x. Thus the columns of coefficients in Figure 1 are also those of Pascal's Triangle, so that the generating function $g_k(x)$ for the k^{th} column is

$$g_k(x) = (1 - x)^{-k} .$$

Using Gould's technique, we first find that the generating function for the k^{th} column with the coefficient of x^r multiplied by F_r is

$$
\begin{aligned}
&[(1 - \alpha x)^{-k} - (1 - \beta x)^{-k}]/(\alpha - \beta) \\
(2) \qquad &= \left[\sum_{j=0}^{k} (-1)^{j+1}\binom{k}{j}(\alpha^j - \beta^j)x^j\right] \Bigg/ (1 - x - x^2)^k(\alpha - \beta) \\
&= \left[\sum_{j=0}^{k} (-1)^{j+1}\binom{k}{j}F_j x^j\right] \Bigg/ (1 - x - x^2)^k ,
\end{aligned}
$$

where

$$\alpha = (1 + \sqrt{5})/2, \qquad \beta = (1 - \sqrt{5})/2 .$$

We then make all exponents corresponding to coefficients of $f_n(x)$ to be n^{-1} by multiplying the above by x^{2k-1}, which gives the row adjusted generating function for the k^{th} column to be

$$h_k(x) = \frac{x^{2k-1}}{\alpha - \beta}\left[(1 - \alpha x)^{-k} - (1 - \beta x)^{-k}\right] .$$

Then

$$
\begin{aligned}
G(x) &= \sum_{n=1}^{\infty} a_n x^n = \sum_{k=1}^{\infty} h_k(x) = \frac{x^{-1}}{\alpha - \beta}\left[\sum_{k=1}^{\infty}\left(\frac{x^2}{1 - \alpha x}\right)^k - \sum_{k=1}^{\infty}\left(\frac{x^2}{1 - \beta x}\right)^k\right] \\
(3) \qquad &= \frac{x^{-1}}{\alpha - \beta}\left[\frac{\dfrac{x^2}{1 - \alpha x}}{1 - \dfrac{x^2}{1 - \alpha x}} - \frac{\dfrac{x^2}{1 - \beta x}}{1 - \dfrac{x^2}{1 - \beta x}}\right] = \frac{x^2}{1 - x - 3x^2 + x^3 + x^4} .
\end{aligned}
$$

Result (3) also follows from Problem H-51 [9]. This states that

$$\sum_{k=1}^{\infty} Q_k(x)t^k = \frac{xt}{1-(2-x)t+(1-x-x^2)t^2} ,$$

where

$$Q_k(x) = \sum_{j=0}^{k} (-1)^{j+1}\binom{k}{j} F_j x^j .$$

Thus using (2),

$$G(x) = \sum_{k=1}^{\infty} h_k(x) = x^{-1}\sum_{k=1}^{\infty} Q_k(x)\left(\frac{x^2}{1-x-x^2}\right)^k$$

$$= \frac{x^2}{1-x-3x^2+x^3+x^4} .$$

The auxiliary polynomial for the recurrence relation obeyed by the a_n is therefore

(4) $$y^4 - y^3 - 3y^2 + y + 1.$$

The second method uses the generating function for $f_n(t)$. Zeitlin [10] has shown that

$$H(x, t) = \frac{x}{1-tx-x^2} = \sum_{n=0}^{\infty} f_n(t)x^n .$$

Since

$$a_n = [f_n(\alpha) - f_n(\beta)]/(\alpha - \beta) ,$$

$$\begin{aligned} G(x) &= \frac{H(x,\alpha) - H(x,\beta)}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta}\left[\frac{x}{1 - \alpha x - x^2} - \frac{x}{1 - \beta x - x^2}\right] \\ &= \frac{x^2}{1 - x - 3x^2 + x^3 + x^4} \end{aligned}$$

the same as (3).

The third method suggested to the authors by Kathleen Weland, varies the pattern in Figure 1. Write the Fibonacci polynomials as in Figure 2. Then it follows from (1) that the generating function in powers of y for the k^{th} column

$$\begin{array}{cccccccccc} & & & & & & & & & 1 \\ & & & & & & & x & & \\ & & & & & x^2 & & & & +1 \\ & & & x^3 & & & & +2x & & \\ & x^4 & & & & +3x^2 & & & & +1 \\ x^5 & & & +4x^3 & & & & +3x & & \\ & & & \vdots & & & & & & \end{array}$$

Figure 2

from the right is

$$\frac{x^k y^{k+1}}{(1 - y^2)^{k+1}} ,$$

where powers of y for terms on the same row are equal. Then multiplying the k^{th} column by F_k, putting $x = 1$, and summing gives

$$G(y) = \frac{y}{1 - y^2} \sum_{k=0}^{\infty} F_k \left(\frac{y}{1 - y^2}\right)^k .$$

Now

$$\sum_{k=0}^{\infty} F_k z^k = \frac{z}{1 - z - z^2}$$

so that

$$G(y) = \frac{y^2}{1 - y - 3y^2 + y^3 + y^4},$$

agreeing with our (3).

Our fourth method uses a matrix approach. It follows by induction that if

$$R(x) = \begin{pmatrix} x & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$R^n(x) = \begin{pmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{pmatrix} \quad (n \geq 0) .$$

Since $f_n(1) = F_n$, we have

$$R^n(1) = Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} .$$

Then the upper right corner of $f_n(Q)$ is a_n. Letting

$$\overline{R}(Q) = \left(\begin{array}{c|c} Q & I \\ \hline I & 0 \end{array}\right),$$

where I is the identity matrix and 0 is the zero matrix, since we may multiply partitioned matrices by blocks, we then have

$$\overline{R}^n(Q) = \left(\begin{array}{c|c} f_{n+1}(Q) & f_n(Q) \\ \hline f_n(Q) & f_{n-1}(Q) \end{array}\right).$$

By the Cayley-Hamilton Theorem, $\overline{R}(Q)$ satisfies its own characteristic polynomial $p(x)$. Since a_n is one of the entries of $\overline{R}^n(Q)$, it obeys a recurrence relation whose auxiliary polynomial is $p(x)$. The desired polynomial is thus

$$p(x) = \det[xI - \overline{R}(Q)] = \det \begin{pmatrix} x-1 & -1 & -1 & 0 \\ -1 & x & 0 & -1 \\ -1 & 0 & x & 0 \\ 0 & -1 & 0 & x \end{pmatrix} \tag{5}$$

$$= x^4 - x^3 - 3x^2 + x + 1 ,$$

which agrees with (4).

A slight extension of the second method will handle second-order recurrent sequences. A generalization of the matrix method will be described later, and the most general solution to our problem, based on the second method, will be given in the last section.

Let W_n obey

$$W_{n+2} = pW_{n+1} - qW_n, \quad p^2 - 4q \neq 0 ,$$

and let $a \neq b$ satisfy

$$x^2 - px + q = 0 .$$

Then

$$a + b = p, \quad ab = q ,$$

and there are constants C and D such that

$$W_n = Ca^n + Db^n$$

for all values of n. We consider the sequence

$$c_n = f_n(W), \quad W^k \equiv W_k .$$

It is easily seen that

$$c_n = Cf_n(a) + Df_n(b) ,$$

implying

$$K(x) = \sum_{n=0}^{\infty} c_n x^n = C\,H(x,a) + D\,H(x,b)$$

$$= \frac{C}{1 - ax - x^2} + \frac{D}{1 - bx - x^2}$$

$$= \frac{(C + D)(1 - x^2) - ab(Ca^{-1} + Db^{-1})x}{1 - px + (q - 2)x^2 + px^3 + x^4}$$

$$= \frac{W_0(1 - x^2) - qxW_{-1}}{1 - px + (q - 2)x^2 + px^3 + x^4} \tag{6}$$

Putting

$$p = 1, \quad q = -1, \quad W_0 = 0, \quad W_1 = 1$$

makes $W_n = F_n$, and $K(x)$ reduces to $G(x)$.

3. A PROPERTY OF 2-BY-2 BLOCK DETERMINANTS

If, in the previous section, we had evaluated

$$\det[xI - \overline{R}(Q)] = \det\left(\begin{array}{c|c} xI - Q & -I \\ \hline -I & xI \end{array}\right)$$

by formally expanding the right side as a usual determinant and taking the determinant of the result, we would have obtained the correct answer; that is,

$$\det\left(\begin{array}{c|c} xI - Q & -I \\ \hline -I & xI \end{array}\right) = \det\,(x^2 I - xQ - I^2)\,.$$

We shall encounter such types of 2-by-2 block determinants while generalizing the matrix approach to symbolic substitutions, so it is convenient to state the following

Theorem: Let $A = (a_{ij})$ and $B = (b_{ij})$ $(i, j = 1, \cdots, n)$ by any n-by-n matrices. Then

$$D(k, m) = \det\left(\begin{array}{c|c} A & mI \\ \hline kI & B \end{array}\right) = \det\,(AB - kmI)\,, \tag{7}$$

where k and m are any real constants.

<u>Proof.</u> The result is familiar when $k = 0$ [4, Section 5.4]. Then assume $k \neq 0$, and consider

$$D(k, m) = \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & m & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 & m & \cdots & 0 \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & & m \\ k & 0 & \cdots & 0 & b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & k & \cdots & 0 & b_{21} & b_{22} & \cdots & b_{2n} \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ 0 & 0 & \cdots & k & b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

We eliminate the bottom row by multiplying the n^{th} column by b_{nj}/k and subtracting from the $(n + j)^{\text{th}}$ column for $j = 1, \cdots, n$, and expanding along the bottom row to yield

$$k(-1)^n \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & m - a_{n1}b_{n1}/k & \cdots & -a_{1n}b_{nn}/k \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & -a_{nn}b_{n1}/k & \cdots & m - a_{nn}b_{nn}/k \\ k & 0 & \cdots & 0 & b_{11} & \cdots & b_{1n} \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & k & b_{n-1,1} & \cdots & b_{n-1,n} \end{pmatrix}.$$

Repeating this process of elimination on the resulting bottom row for $n - 1$ more times gives

$$D(k,m) = k^n(-1)^{n^2} \det \begin{pmatrix} m - \sum_{j=1}^{n} a_{1j}b_{j1}/k & \cdots & - \sum_{j=1}^{n} a_{1j}b_{jn}/k \\ \vdots & & \vdots \\ - \sum_{j=1}^{n} a_{nj}b_{j1}/k & \cdots & m - \sum_{j=1}^{n} a_{nj}b_{jn}/k \end{pmatrix}$$

Now $(-1)^{n^2} = (-1)^n$, and for an n-by-n matrix M,

$$(-k)^n \det M = \det(-kM) ,$$

so that

$$D(k,m) = \det(AB - kmI) .$$

A slightly more general form of the above Theorem was located as a problem in [4, Section 5.4].

4. A GENERALIZED MATRIX METHOD FOR SYMBOLIC SUBSTITUTIONS

We shall now extend the matrix technique used in Section 2. Given arbitrary matrices A and B of the same square dimension, let the $(r,s)^{th}$ entry b_n of B^nA be the n^{th} member of the sequence $\{b_n\}$. We find the auxiliary polynomial for the recurrence obeyed by

$$d_n = f_n(b), \quad b^k \equiv b_k .$$

Clearly the $(r,s)^{th}$ entry of $f_n(B)A$ is d_n. We also have

$$\overline{R}^n(B) \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right) = \left(\begin{array}{c|c} f_{n+1}(B)A & f_n(B)A \\ \hline f_n(B)A & f_{n-1}(B)A \end{array}\right) .$$

It follows that the sequence $\{d_n\}$ obeys a recurrence relation whose auxiliary polynomial is the characteristic polynomial $p(x)$ of $\overline{R}(B)$. Using (7), the latter becomes

$$p(x) = \det[xI - \overline{R}(B)] = \det\left(\begin{array}{c|c} xI - B & -I \\ \hline -I & xI \end{array}\right) \tag{8}$$

$$= \det[(x^2 - 1)I - xB] .$$

The following are some particular cases of this result.

(i) Substitution of Fibonacci numbers. For $B = Q$, as defined above, we obtain (5).

(ii) Substitution of second-order recurrent sequences. Let W_n be as defined in Section 2, and let

$$A = \begin{pmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{pmatrix}, \quad B = \begin{pmatrix} p & -q \\ 1 & 0 \end{pmatrix} .$$

Then

$$B^nA = \begin{pmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{pmatrix},$$

so letting $r = 1$, $s = 2$, we have $b_n = W_n$. In this case

$$p(x) = \det\begin{pmatrix} x^2 - 1 - px & qx \\ -x & x^2 - 1 \end{pmatrix}$$

$$= x^4 - px^3 + (q - 2)x^2 + px + 1 ,$$

agreeing with (6).

(iii). Substitution of Fibonacci polynomials. There is nothing to restrict $\{b_n\}$ itself from being a sequence of polynomials. To illustrate this, put $A = I$ and $B = R(t)$, so that if we let b_n be the upper right term of B^nA, $b_n = f_n(t)$. Then the sequence

$$f_n[f(t)] , \quad f^k(t) \equiv f_k(t) ,$$

obtained by symbolically substituting the Fibonacci polynomials $f_n(t)$ into the Fibonacci polynomials obeys a recurrence relation whose auxiliary polynomial is

$$\det\left[(x^2 - 1)I - xR(t)\right] = x^4 - tx^3 - 3x^2 + tx + 1.$$

(iv) Substitution of Fibonacci numbers with subscripts in an arithmetic progression. Let the sequence $\{r_n\}$ be generated by

$$r_n = F^s f_n(F^k), \quad F^m \equiv F_m .$$

that is, the sequence is formed by replacing x^n by F_{nk+s} in the Fibonacci polynomials. Now $y_n = F_{nk+s}$ obeys

$$y_{n+2} = L_k y_{n+1} - (-1)^k y_n .$$

Applying (ii), with $p = L_k$ and $q = (-1)^k$, we see that the required auxiliary polynomial is

$$x^4 - L_k x^3 + \left[(-1)^k - 2\right] x^2 + L_k x + 1 .$$

(v) Substitution of powers of the integers. Let $e_n(k) = e_n = n^k$ for fixed $k \geq 0$. We find the auxiliary polynomial of the recurrence obeyed by

$$g_n = f_n(e), \quad e^m \equiv e_m .$$

It is easy to show by induction that

$$B_1^n = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix},$$

and in general that

$$B_r^n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \cdot & & & & \cdot \\ \binom{r}{r} & \binom{r}{r-1} & & \cdots & \binom{r}{0} \end{pmatrix}^n = \begin{pmatrix} 1 & 0 & & \cdots & 0 \\ n & 1 & & \cdots & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \binom{r}{r}n^r & \binom{r}{r-1}n^{r-1} & & \cdots & \binom{r}{0} \end{pmatrix}.$$

Then the lower left term of $f_n(B_k)$ is g_n. The desired polynomial is thus

$$p(x) = \det[(x^2 - 1)I - xB_k] = \det \begin{pmatrix} x^2 - x - 1 & 0 & \cdots & 0 \\ & x^2 - x - 1 & \cdots & 0 \\ & & \ddots & \\ \star & & & x^2 - x - 1 \end{pmatrix}$$

$$= (x^2 - x - 1)^{k+1} ,$$

where the $\star$ indicates irrevelant terms. Notice that when $k = 0$ the auxiliary polynomial is $x^2 - x - 1$, which agrees with $f_n^{(1)} = F_n$.

(vi) Substitution of powers of Fibonacci numbers. Let $v_k = F_k^m$, m a fixed integer. Consider

$$h_n = f_n(v), \quad v^k \equiv v_k .$$

We require a matrix whose n^{th} power has F_n^m as an entry. Such a matrix is provided by Problem H-26 [8]. Let $B_m = (b_{rs})$, where

$$b_{rs} = \binom{r-1}{m+1-s}$$

for $r, s = 1, \cdots, m+1$. Then putting $r = m+1$, $s = 1$, we have that the $(r,s)^{th}$ entry of B_m^n is indeed F_n^m. Thus the $(r,s)^{th}$ entry of $f_n(B_m)$ is h_n, and in this case the auxiliary polynomial is

$$p(x) = \det[(x^2 - 1)I - xB_m] = x^{m+1} \det\left[\left(\frac{x^2 - 1}{x}\right)I - B_m\right].$$

Put $(x^2 - 1)/x = y$. Now $\det(yI - B_m)$ has been evaluated [1;2;5] to be

$$\det(yI - B_m) = \sum_{r=0}^{m+1} (-1)^{\frac{(m-r)(m-r+1)}{2}} \begin{bmatrix} m+1 \\ r \end{bmatrix} y^r ,$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix} = \frac{F_m F_{m-1} \cdots F_{m-r+1}}{F_1 F_2 \cdots F_r} \quad (r > 0); \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1 ,$$

Now

$$y^2 = x^{-r}(x^2 - 1)^r = \sum_{j=0}^{r} (-1)^j \binom{r}{j} x^{r-2j} ,$$

so that

$$(9) \qquad p(x) = \sum_{r=0}^{m+1} \sum_{j=0}^{r} (-1)^{j+(m-r)(m-r+1)/2} \begin{bmatrix} m+1 \\ r \end{bmatrix} \binom{r}{j} x^{m+r-2j+1}$$

$$= \sum_{s=0}^{2m+2} \left[\sum_{r=0}^{m+1} (-1)^{[(m-r)(m-r+1)+s-m-r-1]/2} \begin{bmatrix} m+1 \\ r \end{bmatrix} \binom{r}{\{s-m-r-1\}/2} \right] x^s ,$$

where in the last expression the summand is zero if $(s - m - r - 1)/2$ is not an integer.

This result may be extended to powers of an arbitrary second-order recurrent sequence $\{W_n\}$, described in Section 2, by using the matrix $C_m = (c_{rs})$, where

$$c_{rs} = \binom{r-1}{m+1-s} p^{r+s-m} q^{m+1-r}$$

for $r, s = 1, \cdots, m + 1$. For a discussion of C_m see [5]. Letting $u_n = (a^n - b^n)/(a - b)$, where a and b are as in Section 2, define

$$\begin{bmatrix} m \\ r \end{bmatrix}_u = \frac{u_m u_{m-1} \cdots u_{m-r+1}}{u_1 u_2 \cdots u_r} \quad (r > 0) \quad \begin{bmatrix} m \\ 0 \end{bmatrix}_u = 1 .$$

Then the counterpart to (9) is

$$p(x) = \sum_{r=0}^{m+1} \sum_{j=0}^{r} (-1)^{m+1-r-j} (-q)^{(m-r+1)(m-r+2)/2} \begin{bmatrix} m+1 \\ r \end{bmatrix}_u \binom{r}{j} x^{m+r-2j+1} . \tag{10}$$

In particular, (10) is the auxiliary polynomial for the recurrence relation obeyed by the symbolic substitution of $\{F^m_{nk+s}\}$ for proper choices of the parameters.

The matrix method developed here is more general than previously indicated. In particular, the full power of (7) has not been exploited. For example, let $\{p_n(x)\}$ be any sequence of polynomials (numbers) obeying

$$p_{n+2}(x) = g(x)p_{n+1}(x) + hp_n(x) , \tag{11}$$

where $g(x)$ is any polynomial in x independent of n, and h is a real constant. Let the sequence $\{b_n\}$ be generated by the matrices A and B as before. We shall find the auxiliary polynomial of the recurrence relation obeyed by

$$s_n = p_n(b), \qquad b^k \equiv b_k .$$

Now the $(r, s)^{th}$ entry of $p_n(B)A$ is s_n. Also, if

$$F(B) = \left(\begin{array}{c|c} g(B) & hI \\ \hline I & 0 \end{array}\right), \quad G(B) = \left(\begin{array}{c|c} p_2(B) & p_1(B) \\ \hline p_1(B) & p_0(B) \end{array}\right) ,$$

then

$$F^n(B)G(B)\left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right) = \left(\begin{array}{c|c} p_{n+2}(B)A & p_{n+1}(B)A \\ \hline p_{n+1}(B)A & p_n(B)A \end{array}\right),$$

Since s_n is an entry in the right-hand matrix, it follows that the sequence $\{s_n\}$ obeys a recurrence relation whose auxiliary polynomial is the characteristic polynomial of $F(B)$. Using (7), the latter reduces to

$$\begin{aligned} \det[xI - F(B)] &= \det\left(\begin{array}{c|c} xI - g(B) & -hI \\ \hline -I & xI \end{array}\right) \\ &= \det[(x^2 - h)I - xg(B)] . \end{aligned}$$

Putting $g(x) = x$, $h = 1$, $p_1(x) = 1$, and $p_2(x) = x$ specializes this to (8). As another illustration of this result, we note that $T_n(x)$ and $U_n(x)$, the Chebyshev polynomials of the first and second kind, respectively, obey (11) for $g(x) = 2x$, $h = -1$, along with

$$T_0(x) = 1 = U_0(x), \quad T_1(x) = x, \quad \text{and} \quad U_1(x) = 2x .$$

Then the sequences defined by the symbolic substitutions

$$T_n(F), \quad U_n(F), \quad F^k \equiv F_k ,$$

each obey a recurrence relation whose auxiliary polynomial is

$$\det[(x^2 + 1)I - 2xQ] = x^4 - 2x^3 - 2x^2 - 2x + 1 . \tag{12}$$

5. A GENERAL RESULT

Here we extend the second approach in Section 2 to obtain the most general solution to our problem. Let $\{q_n(x)\}$ be any sequence of polynomials obeying the k^{th} order recurrence relation

$$0 = \sum_{j=0}^{k} a_j(x)q_{n-j}(x), \quad a_0(x)a_k(x) \neq 0 ,$$

where the $a_j(x)$ are polynomials independent of n. Put

$$Q(x,t) = \sum_{j=0}^{k} a_j(x)t^j ,$$

so that

$$M(x,t) = \sum_{n=0}^{\infty} q_n(x)t^n = \frac{P(x,t)}{Q(x,t)} ,$$

where $P(x,t)$ is a polynomial in x and t of degree $<k$ in t. Suppose $\{A_n\}$ is a sequence satisfying an m^{th} order recurrence relation with constant coefficients whose auxiliary polynomial has distinct roots $r_1, r_2, \cdots, r_m$. Then there exist constants $B_1, B_2, \cdots, B_m$ such that

$$A_n = \sum_{i=1}^{m} B_i r_i^n .$$

Define $\{D_n\}$ by

$$D_n = q_n(A), \quad A^k \equiv A_k .$$

Then

$$D_n = \sum_{i=1}^{m} B_i q_n(r_i) ,$$

so that

$$\sum_{n=0}^{\infty} D_n t^n = \sum_{i=1}^{m} B_i M(r_i, t) = \sum_{i=1}^{m} \frac{B_i P(r_i, t)}{Q(r_i, t)} = \frac{r(t)}{Q(r_1, t) \cdots Q(r_m, t)} = \frac{r(t)}{s(t)} ,$$

where the degree of $r(t) < mk$, and the degree of $s(t) = mk$. Therefore $\{D_n\}$ obeys a recurrence relation whose auxiliary polynomial is

$$t^{mk} s(1/t) = \prod_{i=1}^{m} \left[\sum_{j=0}^{k} a_j(r_i) t^{k-j} \right] . \tag{13}$$

Continuing with the illustration of the preceding section, for Chebyshev polynomials of both kinds we have

$$k = 2, \quad a_0(x) = a_2(x) = 1, \quad a_1(x) = -2x ,$$

and if $A_n = F_n$ we see

$$m = 2, \quad r_1 = (1 + \sqrt{5})/2, \quad r_2 = (1 - \sqrt{5})/2 .$$

The desired polynomial is then

$$(t^2 - 2r_1 t + 1)(t^2 - 2r_2 t + 1) = t^4 - 2t^3 - 2t^2 - 2t + 1 ,$$

in agreement with (12).

It happens that (13) is valid even if $r_1, \cdots, r_m$ are not distinct. Then this generalization actually yields the matrix method as a special case. To see this, put

$$k = 2, \quad a_0(x) = 1, \quad a_1(x) = -g(x), \quad a_2(x) = -h,$$

and let b_n be the $(r,s)^{th}$ entry of $B^n A$, where A and B are m-by-m matrices. Then $\{b_n\}$ obeys a recurrence relation whose auxiliary polynomial is

$$\det(xI - B) = (x - r_1) \cdots (x - r_m) .$$

From (13), we have that the sequence

$$\{q_n(b)\},\ b^k \equiv b_k \quad ,$$

obeys a recurrence relation whose auxiliary polynomial is

$$p_1(t) = \prod_{i=1}^{m} \left[t^2 - g(r_i)t - h \right].$$

On the other hand, by (8) we find that the matrix method gives the required polynomial as

$$p_2(t) = \det \left[(t^2 - h) - g(b)t \right] .$$

To show $p_1(t) = p_2(t)$, we note B is similar to

$$C = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ & r_2 & \cdots & 0 \\ \star & & \ddots & \\ & & & r_m \end{pmatrix} ,$$

so that $g(B)$ is similar to $g(C)$. We also have

$$g(C) = \begin{pmatrix} g(r_1) & 0 & \cdots & 0 \\ & g(r_2) & \cdots & 0 \\ \star & & \ddots & \\ & & & g(r_m) \end{pmatrix} ,$$

where the $\star$ indicates irrevelant entries. Since similar matrices have the same characteristic polynomial,

$$p_2(t) = \det \left[(t^2 - h) - g(C)t \right] = \prod_{i=1}^{m} \left[t^2 - h - tg(r_i) \right] = p_1(t) .$$

However, the matrix method has the advantage that the roots $r_1, \cdots, r_m$ of the characteristic polynomial of B do not have to be known.

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REFERENCES

1. Terrence A. Brennan, "Fibonacci Powers and Pascal's Triangle in a Matrix," Fibonacci Quarterly 2 (1964), No. 2, pp. 93-103.
2. L. Carlitz, "The Characteristic Polynomial of a Certain Matrix of Binomial Coefficients," Fibonacci Quarterly, 3 (1965), No. 2, pp. 81-89.
3. H. W. Gould, "Generating Functions for Products of Powers of Fibonacci Numbers," Fibonacci Quarterly, 1 (1963), No. 2, pp. 1-16.
4. K. Hoffman and R. Knuze, Linear Algebra, Prentice-Hall, Englewood Cliffs, N. J., 1961.
5. V. E. Hoggatt, Jr., and A. P. Hillman, "The Characteristic Polynomial of the Generalized Shift Matrix," Fibonacci Quarterly, 3 (1965), No. 2, p. 91.
6. E. D. Rainville, Special Functions, MacMillan Co., New York, 1960.
7. Problem H-18, proposed by R. G. Buschman, Fibonacci Quarterly, 1 (1963), No. 2, p. 55; Solution, 2 (1964), No. 2, pp. 126-129.
8. Problem H-26, proposed by L. Carlitz, Fibonacci Quarterly, 1 (1963), No. 4, pp. 47-48; Solution to corrected version, 3 (1965), No. 3, pp. 205-207; Addition to solution, 3 (1965), No. 4, p. 302.
9. Problem H-51, proposed by L. Carlitz and V. E. Hoggatt, Jr., Fibonacci Quarterly, 2 (1964), No. 4, p. 304.
10. Problem B-74, proposed by M. N. S. Swamy, Fibonacci Quarterly, 3 (1965), No. 3, p. 236; Solution, 4 (1966), No. 1, pp. 94-96.

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