FIBONACCI SEQUENCES WITH IDENTICAL CHARACTERISTIC VALUES

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A Fibonacci sequence is viewed herein as an integer sequence

$$\{f_n\}_{n=-\infty}^{\infty}$$

which satisfies the recursion

$$(1) \qquad f_n = f_{n-1} + f_{n-2}$$

for all n.

Following [1], it is convenient to associate two Fibonacci sequences with each other if one can be transformed into the other by a relabeling of indices. Also, it is apparent that $\{f_n\}$ satisfying (1) implies that $\{-f_n\}$ satisfies (1) and it is convenient to associate a sequence with its negative. These remarks lead to

Definition. Two Fibonacci sequences $\{f_n\}$ and $\{g_n\}$ are equivalent if and only if there exists an integer k such that either

(i) $g_n = f_{n+k}$ for all n;

or

(ii) $g_n = -f_{n+k}$ for all n.

In [1], the discussion pertains to Fibonacci sequences such that there is no common divisor $d > 1$ of every term in the sequence (or equivalently, of any two consecutive terms). In this paper, we will be interested in all integer sequences satisfying (1). However, when there is no common divisor (>1) of the sequence, we will call the sequence primitive.

A well-known identity satisfied by Fibonacci sequences is

(2) $$f_{n+1}f_{n-1} - f_n^2 = \pm D$$

where $D \geq 0$ and the sign alternates with n. We call the integer

$$D = \left|f_{n+1}f_{n-1} - f_n^2\right|$$

the characteristic of the sequence $\{f_n\}$. The reader may verify that if $\{f_n\}$ is equivalent to $\{g_n\}$ then $\{f_n\}$ and $\{g_n\}$ have the same characteristic.

A table is presented in [1] of all $D < 1000$ for which there exists a primitive sequence. Also, all primitive sequences (up to equivalence) having these characteristics are provided. Such a table leads one to ask the following two questions:

(I) For a given integer $D \geq 0$, how many Fibonacci Sequences are there (up to equivalence) having the characteristic D?

(II) For a given $D \geq 0$, how many primitive Fibonacci sequences are there (up to equivalence) having the characteristic D?

This paper is devoted to providing a complete answer to each of these questions.

For this purpose, we let

$$\alpha = \frac{1+\sqrt{5}}{2}$$

and we consider the field extension $R(\alpha)$ obtained by adjoining α to the rationals. The domain of algebraic integers in $R(\alpha)$ then consists of all numbers of the form $A + B\alpha$, where A and B are rational integers. It is well known (see [2]) that one has unique factorization in this domain of integers. The units in this domain are precisely numbers of the form $\pm\alpha^{\pm n}$ and all primes (up to associates) are

(i) $\sqrt{5} = 2\alpha - 1$

(ii) all rational primes of the form $5k \pm 2$

(iii) numbers of the form $A + B\alpha$ and $A + B\bar{\alpha}$, where $\bar{\alpha}$ is the conjugate of α, i.e.,

$$\bar{\alpha} = \frac{1-\sqrt{5}}{2}$$

and $|(A + B\alpha)(A + B\bar{\alpha})|$ is a rational prime of the form $5k \pm 1$.

We may assign to each Fibonacci sequence an integer ξ in $R(\alpha)$, namely, the sequence $\{f_n\}$ is assigned the integer $\xi = f_0 + f_1\alpha$. It is easily verified that the assignment of integers in this manner provides a one-to-one correspondence between Fibonacci sequences and integers in $R(\alpha)$. Letting $\xi = A + B\alpha$ be an integer in $R(\alpha)$ we denote by $S(\xi)$ the unique sequence assigned to ξ (i.e., the sequence determined by $f_0 = A$, $f_1 = B$).

The assignment $S(\xi)$ preserves addition in the sense that if $S(\xi_1) = \{f_n\}$ and $S(\xi_2) = \{g_n\}$, then $S(\xi_1 + \xi_2) = \{f_n + g_n\}$. It might also be remarked that the correspondence $S(\xi)$ allows one to define a product of two Fibonacci sequences in a natural way. Namely, for two Fibonacci sequences $S(\xi_1)$ and $S(\xi_2)$, the product sequence is defined as $S(\xi_1\xi_2)$. In this way, one has a ring of Fibonacci sequences which is isomorphic to the ring of integers in $R(\alpha)$.

Two integers ξ_1 and ξ_2 in $R(\alpha)$ are called associates if $\xi_1 = \epsilon\xi_2$ for some unit ϵ (which is one of the integers $\pm\alpha^{\pm n}$). It follows that two sequences $S(\xi_1)$ and $S(\xi_2)$ are equivalent if and only if ξ_1 and ξ_2 are associates.

For a given integer $\xi = A + B\alpha$, we define the (absolute) norm $N(\xi)$ in the usual way as $N(\xi) = |\xi\bar{\xi}|$, where $\bar{\xi} = A + B\bar{\alpha}$. One can easily verify that the characteristic D of a Fibonacci sequence $S(\xi)$ is simply $N(\xi)$.

As a result of the above remarks, we find that questions (I) and (II) reduce to questions about integers in $R(\alpha)$. Namely, (I) and (II) are equivalent to asking:

(Ia) How many integers in $R(\alpha)$ (up to associates) have a given norm D?

(IIa) How many integers in $R(\alpha)$ (up to associates) with no rational integer divisor $d > 1$ have a given norm D?

To resolve these questions we introduce:

$P^* = \{$set of all positive rational integers n such that every prime divisor of n is of the form $5k \pm 1\}$

and by convention 1 belongs to P^*;

$\omega(n)$ = number of distinct prime divisors of n;

$$d_+(n) = \sum_{\substack{d|n,\, d>0 \\ d\equiv\pm1 \pmod 5}} 1 \;; \qquad d_-(n) = \sum_{\substack{d|n,\, d>0 \\ d\equiv\pm2 \pmod 5}} 1 \;;$$

$$r(n) = d_+(n) - d_-(n),$$

where $r(0) = 1$ by convention. (To illustrate, $\omega(60) = 3$, $d_+(60) = 3$, $d_-(60) = 3$, $r(60) = 0$). The answers to (I) and (II) may now be provided in a compact form as follows:

Theorem 1. For $D \geq 0$, there are exactly $r(D)$ Fibonacci sequences (up to equivalence) having characteristic D.

Theorem 2. There exists a primitive sequence having characteristic $D \geq 0$ if and only if $D = n$ or $D = 5n$, where n belongs to $P^\star$. For such a characteristic D, the number of (inequivalent) primitive sequences is exactly $2^{\omega(n)}$.

Proofs: Letting

$$D = 5^a p_1^{b_1} p_2^{b_2} \cdots p_h^{b_h} q_1^{c_1} \cdots q_k^{c_k}$$

be the prime factorization of D, where p_i is a prime of the form $5m \pm 1$ and q_j a prime of the form $5m \pm 2$ it follows that all integers in $R(\alpha)$ having norm D are

$$\text{(3)} \qquad A + B\alpha = \epsilon(\sqrt{5})^a \prod_{i=1}^{h} (A_i + B_i\alpha)^{s_i}(A_i + B_i\bar{\alpha})^{t_i} \prod_{j=1}^{k} q_j^{c_j/2},$$

where ϵ is a unit, $s_i + t_i = b_i$, c_j of necessity is even, and $A_i + B_i\alpha$ is a prime in $R(\alpha)$ having norm p_i. Thus, the number of integers (up to associates) having norm D is the number of ways we can vary each s_i with $0 \leq s_i \leq b_i$. The number of such choices for the s_i is the product $\prod_i (1 + b_i)$. The latter expression (combined with the fact that all c_j must be even) is equivalent to Theorem 1.

This equivalence is a counting exercise which can be ascertained in the following way. The factor 5^a of D has no effect upon the value of $r(D)$. Letting

$$\tilde{D} = \prod_{i=1}^{h} p_i^{b_i} \prod_{j=1}^{k} q_j^{c_j} ,$$

one has $r(D) = r(\tilde{D})$. The divisors of $\tilde{D}$ are the terms in the expansion

$$\prod_{i=1}^{h} (1 + p_i + \cdots + p_i^{b_i}) \prod_{j=1}^{k} (1 + q_j + \cdots + q_j^{c_j}) . \tag{4}$$

By replacing each p_i with the value +1 and each q_j with the value -1 in (4), the resulting expansion will yield a term of +1 for each divisor of the form $5m \pm 1$ and a term of -1 for each divisor of the form $5m \pm 2$. Thus, the expansion of the modified expression is merely $r(\tilde{D})$. If any c_j is odd the factor $(1 + (-1) + \cdots + (-1)^{c_j})$ is zero which yields $r(\tilde{D}) = 0$. If all c_j are even, then the factor corresponding to q_j is $(1 + (-1) + \cdots + (-1)^{c_j}) = 1$ and the resulting expression for $r(\tilde{D})$ becomes $\prod_i (1 + b_i)$ which is the desired result.

Theorem 2 is obtained by realizing that for (3) to have no rational integer divisor (>1), one must have $a = 0$ or 1, $c_j = 0$ for all j, and the only choices for s_i are 0 and b_i. Thus, there are 2^k choices for s_i, which is theorem 2.

As a final note, it should be pointed out that the proofs of Theorems 1 and 2 proceed in a manner analogous to that which one could take in determining the number of representations of an integer N as the sum of two squares (see Theorem 278 of [2]). In this latter problem one utilizes the ring of gaussian integers whereas in the problems considered above we have relied upon the ring of integers in $R(\alpha)$. It would appear that the above results should extend to other recursions of the form $f_n = af_{n-1} + bf_{n-2}$ provided one has unique factorization in the underlying ring of integers.

For a related paper, see Thoro [3].

REFERENCES

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2. Hardy and Wright, An Introduction to the Theory of Numbers, Third Edition, 1954, Oxford University Press, pp. 221-223, pp. 240-243.
3. D. E. Thoro, "Application of Unimodular Matrices," Fibonacci Quarterly, Vol. 2, No. 4, Dec. 1964, pp. 291-295.

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FIRST QUADRANT GRAPH OF GAUSSIAN PRIMES