ON THE EXISTENCE OF AN INFINITUDE OF COMPOSITE PRIMITIVE DIVISORS OF SECOND-ORDER RECURRING SEQUENCES

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1. INTRODUCTION

Let \( \alpha \neq 0, \beta = 0, |\alpha| > |\beta| \), be any two complex numbers, such that \( \alpha + \beta \) and \( \alpha \beta \) are two relatively prime integers. Then the numbers

\[
D_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} + \alpha^{n-2}\beta + \cdots + \beta^{n-1}, \quad S_n = \alpha^n + \beta^n
\]

are integers, since they are expressed as rational integral symmetric functions of the roots \( \alpha, \beta \) of an algebraic equation

\[
z^2 - (\alpha + \beta)z + \alpha \beta = 0
\]

with integral coefficients with leading coefficient unity. One may readily verify that \{\( D_n \)\} and \{\( S_n \)\} are second-order recurring sequences satisfying the common recursion relation

\[
X_n = (\alpha + \beta)X_{n-1} - \alpha \beta X_{n-2}.
\]

(Since \( D_0 = 0, \ D_1 = 1; \ S_0 = 2, \ S_1 = \alpha + \beta \), the recursion relation again shows that the numbers \( D_n, S_n \) are integers.) One may also easily verify that \( D_{2n} = D_n S_n \).

A divisor \( > 1 \) of \( D_n, \ n > 1 \), is said to be primitive (or: characteristic) if it is relatively prime to any \( D_i \) with \( 1 < i < n \). The greatest primitive divisor of \( D_n \) is denoted by \( D'_n \). A divisor \( > 1 \) of \( S_n, \ n > 1 \), is said to be primitive (or: characteristic) if it is relatively prime to any \( S_i \) with \( 0 < i < n \). The greatest primitive divisor of \( S_n \) is denoted by \( S'_n \). From \( D_{2n} = D_n S_n \) one may easily deduce that

\[
D_{2n} = S_n.
\]

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For any prime p dividing a certain \( D_i \) with \( i > 1 \), a \( \phi(p) \) denotes the
smallest positive subscript \( n \), such that \( p|D_n \). Thus \( p \) is a primitive divisor
of \( D_{\phi(p)} \).

By \( F_n \) we denote the product

\[
F_n = \prod_{d|n} \mu\left(\frac{n}{d}\right)
\]

where \( \mu \) is the Möbius function.

R. D. Carmichael showed in [1] that for any \( n \neq 4, 6, 12 \) there is

\[
D_n^* = F_n
\]

except when \( n = a(p)p^\lambda \), \( p \) being a prime factor of \( D_n \), \( \lambda \geq 1 \), in which case

\[
D_n^* = \frac{1}{p} F_n
\]

He showed furthermore that if \( n = a(p)p^\lambda \), \( \lambda > 1 \), then \( p \) is the greatest di-
visor of \( n \), except when \( p = 2 \), and \( a(p) = 3 \).

Furthermore Carmichael showed, for \( \alpha, \beta \) real, the following inequalities

\[
\alpha^{\phi(n)-2^{\omega(n)}} < F_n < \alpha^{\phi(n)+2^{\omega(n)}}
\]

where \( \phi \) is Euler's totient function, and \( \omega(n) \) is the number of distinct prime
factors of \( n \).

The main result achieved by Carmichael is the following

**Theorem XXIII.** If \( \alpha \) and \( \beta \) are real and \( n \neq 1, 2, 6 \), then \( D_n \) con-
tains at least one characteristic factor, except when \( n = 12 \), \( \alpha + \beta = \pm 1 \), \( \alpha \beta = -1 \).

In the present paper the above Carmichael's results are generalized for
any two complex numbers \( \alpha \neq 0 \), \( \beta \neq 0 \), \( |\alpha| > |\beta| \), such that \( \alpha + \beta \) and \( \alpha \beta \)
are two relatively prime integers. (However, the exact value of \( n \) beginning
with which any \( D_n \) contains at least one characteristic factor, is not calculated.
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Furthermore, starting from (2), we deduce an asymptotic formula (6) for $F_n$ which is stronger than the inequalities given by Carmichael. Finally, the method of proof used here is slightly simpler than the one used by Carmichael. The main results proved here are the existence of an infinitude of composite $D'_n$ for any $\alpha, \beta$; of composite $D_{2n}$ for $\alpha \neq \emptyset$; and of composite $D_{2n+1}$ for $(\alpha - \beta)^2 \neq 0$, or $(\alpha - \beta)^2 = 0$ and $\alpha \neq 0$.

2. ASYMPOTIC FORMULA FOR $D'_n$

By (2)

$$\log F_n = \sum_{d|n} \mu \left( \frac{n}{d} \right) \log D_d = \sum_{d|n} \mu \left( \frac{n}{d} \right) \log \frac{\alpha^d - \beta^d}{\alpha - \beta} = \log \alpha \sum_{d|n} \mu \left( \frac{n}{d} \right) d$$

$$+ \sum_{d|n} \mu \left( \frac{n}{d} \right) \log \left\{ 1 - \left( \frac{\beta}{\alpha} \right)^d \right\} - \log (\alpha - \beta) \sum_{d|n} \mu \left( \frac{n}{d} \right).$$

Noting that

$$\sum_{d|n} \mu \left( \frac{n}{d} \right) d = \phi(n),$$

and

$$\sum_{d|n} \mu \left( \frac{n}{d} \right) = 0$$

for any $n > 1$, we get

$$\log F_n = \log \alpha \cdot \phi(n) + \sum_{d|n} \mu \left( \frac{n}{d} \right) \log \left\{ 1 - \left( \frac{\beta}{\alpha} \right)^d \right\},$$

for any $n > 1$. 

(5)
Let us evaluate

\[ \sum_{d|n} \mu\left(\frac{n}{d}\right) \log \left\{ 1 - \left(\frac{\beta}{\alpha}\right)^d \right\} . \]

Note that for any \(0 < q < 1\) there exists a positive constant \(A\), such that, for any complex \(z\), for which \(|z| \leq q\), there is

\[ |\log (1 + z)| \leq A|z| , \]

where by \(\log (1 + z)\) the principal value of \(\log\) is understood. Indeed,

\[ \frac{\log (1 + z)}{z} = 1 - \frac{z}{2} + \frac{z^2}{3} - \cdots \]

is an analytic function in the circle \(|z - 1| < q < 1\), hence it is bounded there.

Now, putting \(q = \left|\frac{\beta}{\alpha}\right|\), we have, for any \(d > 1, \left|\frac{\beta}{\alpha}\right| \leq q\). Hence

\[ \left| \sum_{d|n} \mu\left(\frac{n}{d}\right) \log \left\{ 1 - \left(\frac{\beta}{\alpha}\right)^d \right\} \right| \leq \sum_{d|n} \left| \log \left\{ 1 - \left(\frac{\beta}{\alpha}\right)^d \right\} \right| < \sum_{d=1}^{\infty} \left| \log \left\{ 1 - \left(\frac{\beta}{\alpha}\right)^d \right\} \right| \]

\[ < A \sum_{d=1}^{\infty} \left|\frac{\beta}{\alpha}\right|^d = A\left|\frac{\beta}{\alpha}\right| \frac{1}{1 - \left|\frac{\beta}{\alpha}\right|} = A\left|\frac{\beta}{\alpha}\right| \frac{1}{1 - \left|\frac{\beta}{\alpha}\right|} = B , \]

where \(B\) is a positive constant.

Hence, by (5) it follows that

\[ \log F_n = \log \alpha \cdot \phi(n) + o(1) . \]

Now, by (3), (4), we have the following

**Theorem 1.** There is

\[ \log D'_n = \log \alpha \cdot \phi(n) + o(1) , \]
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except when \( n = a(p)\pi^\lambda, \quad \lambda \geq 1, \quad p \) being a prime factor of \( D_n \), in which case it is

(8) \[
\log D_n = \log \alpha \cdot \phi(n) - \log p + O(1).
\]

Now, by assumption, \( \alpha \beta \) is an integer, and \( |\alpha| > |\beta| \), therefore

\[
|\alpha|^2 > |\alpha| \cdot |\beta| = |\alpha \beta| = 1,
\]

hence

\[
|\alpha| > 1, \quad |\log \alpha| \geq |\log |\alpha| > 0.
\]

By a theorem in [2], p. 114, there exists a positive constant \( C \), such that

\[
\phi(n) > \frac{C \cdot n}{\log \log n} \quad \text{for } n > 3.
\]

On the other hand \( p|n \), hence \( \log p \leq \log n \). Hence, by Theorem 1,

(9) \[
\log D_n > |\log \alpha| \cdot \phi(n) - \log p - B > \log |\alpha| \cdot \frac{C \cdot n}{\log \log n} - \log n - B \quad \text{as } n \to \infty,
\]

which means that:

**Theorem 2.** Beginning with a certain positive \( n \), \( D_n \) has at least one primitive factor.

**Remark.** The error term \( O(1) \) in (7) cannot be refined, since if \( n \) is a prime, then

\[
\sum_{\mu(n/d) \log \left(1 - \left(\frac{\beta}{\alpha}\right)\right) = -\log \left(1 - \frac{\beta}{\alpha}\right) + \log \left(1 - \left(\frac{\beta}{\alpha}\right)^n\right) \to -\log \left(1 - \frac{\beta}{\alpha}\right) \neq}
\]
Theorem 3.

\[
\sum_{n=1}^{\infty} \frac{1}{D'_{n}}
\]

converges.

Proof. From (9) it follows that there is a positive constant \( D \) such that, for all \( n \geq 1 \),

\[
D'_n \geq \frac{C \cdot n}{e^{B \cdot n}} \Rightarrow D \cdot n^2
\]

Hence

\[
\sum_{n=1}^{\infty} \frac{1}{D'_{n}} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.
\]

3. MAIN RESULTS

Lemma 1. Be \( N \) the sequence of natural numbers, \( S \) a subsequence of \( N \), and \( A \) a reduced arithmetic progression. Then, an infinitude of \( D'_n \) is composite for

I) \( n \in S \) \quad or \quad II) \( n \in N - S \)

according as

I) \( \) any \quad or \quad II) \( \) no

prime member of \( A \) is a factor of a certain \( D'_{n'} \), \( n \in S \).

Proof. I) Suppose any prime member of \( A \) is a factor of a certain \( D'_{n'} \), \( n \in S \), and that there is a positive integer \( n_0 \) such that any \( D'_{n} \), where \( n \in S \), \( n > n_0 \), is a prime. Let \( q \) be the greatest prime factor of \( D'_{n} \), \( n \leq n_0 \).
Then, by Theorem 3, and noting that

\[ \sum_{p \in A} \frac{1}{p} = \infty, \]

where \( p \) denotes a prime number, we have

\[ \infty > \sum_{n \in \mathbb{N}} \frac{1}{p|n} \approx \sum_{n \in S} \frac{1}{p|n} \approx \sum_{p \in A, p > q} \frac{1}{p} = \infty, \]

whence \( \infty > \infty \), which is absurd. Thus, I) is proved.

II) Suppose no prime member of \( A \) is a factor of a certain \( D'_{n'} \), \( n \in S \). Then, noting that any prime \( p \nmid 2(\alpha - \beta)^2\alpha \beta \) is a factor of a certain \( D'_{n'} \) ([1], p. 45, Theorem XII), any prime member of \( A \) not a factor of \( 2(\alpha - \beta)^2\alpha \beta \) is a factor of a certain \( D'_{n'} \), \( n \in N - S \), and II) follows as above.

**Theorem 4.** There is an infinitude of composite \( D'_{n'} \).

**Proof.** The theorem is an immediate consequence of Lemma 1, noting that any prime \( p \nmid 2(\alpha - \beta)^2\alpha \beta \) is a factor of a certain \( D'_{n'} \).

**Lemma 2.** If \( b \) is an integer, and \( b \neq 0 \), then there exists an odd prime \( p \), such that \( \left( \frac{b}{p} \right) = -1 \), where \( \left( \frac{b}{p} \right) \) is Legendre's symbol. In particular,

I) If \( b = \pm m^2 \), \( \cdots, p_r \), where \( r \geq 1 \) and \( p_1, \cdots, p_r \) are distinct primes, then there exists an integer \( u = 1 \pmod{4} \), where \( (u, 4p_1, \cdots, p_r) = 1 \), such that, for any prime \( p = u \pmod{4p_1, \cdots, p_r} \), it is

\[ \left( \frac{\pm b}{p} \right) = -1. \]

II) If \( b = -m^2 \), then for any prime \( p = -1 \pmod{4} \) it is

\[ \left( \frac{b}{p} \right) = -1. \]

**Proof.** [2], p. 75.
Lemma 3. Let \( p \) be an odd prime. If \( p \mid ax^2 + by^2 \) for some integers \( a, b, x, y, \) and \( p \nmid (x,y) \), then

\[
\left( \frac{-ab}{p} \right) = 1 .
\]

Proof. Since \( p \nmid (x,y) \), \( p \) cannot divide both \( x \) and \( y \). Thus, without loss of generality, we may assume that \( p \nmid x \). Then there exists an integer \( z \), such that \( yz = 1 \pmod{p} \). Hence, from \( ax^2 + by^2 = 0 \pmod{p} \) it follows that

\[
(axz)^2 = -ab \pmod{p},
\]

whence

\[
\left( \frac{-ab}{p} \right) = 1 .
\]

Lemmas 2, 3 imply the following:

Lemma 4. I) If \( b = \pm m^2 p_1 \cdots p_r \), where \( r \geq 1 \) and \( p_1, \cdots, p_r \) are distinct primes, then there exists an integer \( u = 1 \pmod{4} \), where \( (u,4p_1, \cdots, p_r) = 1 \), such that, for any prime \( p = u \pmod{4p_1, \cdots, p_r} \), it is \( p \nmid x^2 + by^2 \) for any integers \( x, y \), such that \( p \nmid (x,y) \).

II) If \( b = m^2 \) and \( p \nmid (x,y) \), then \( p \nmid x^2 + by^2 \) for any prime \( p = -1 \pmod{4} \).

Theorem 5. If \( \alpha \beta \neq -\Box \), then there is an infinitude of composite \( D_n \).

Proof. One may readily verify that

\[ D_{2n+1} = D_{n+1}^2 - \alpha \beta D_n^2 . \]

On the other hand, \( (D_{n+1}', D_n) = 1 \) ([1], p. 38, Corollary). Hence, putting in Lemma 4:

\[ b = -\alpha \beta, \quad x = D_{n+1}', \quad y = D_n , \]

and noting that, according to the assumption, \( b = -\alpha \beta \neq -\Box \), there exists a reduced arithmetic progression \( A \), no prime member of which divides \( D_{2n+1} \).
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Hence, no prime member of $A$ is a factor of $D_{2n+1}$. The theorem follows by
Lemma 1, II).

Theorem 6. If

I) \[ (\alpha - \beta)^2 \neq \pm 1, \]

or

II) \[ (\alpha - \beta)^2 = \pm 1 \text{ and } \alpha \beta \neq \pm 1, \]

then there is an infinitude of composite $D_{2n+1}$.

Proof. One may readily verify that

(9) \[ S_n^2 = (\alpha - \beta)^2 D_n^2 + 4(\alpha \beta)^n. \]

I) Suppose that $(\alpha - \beta)^2 \neq \pm 1$. Then $(\alpha - \beta)^2 = \pm m^2 p_1, \ldots, p_r$, where $r \geq 1$ and $p_1, \ldots, p_r$ are distinct primes. Then, by Lemma 2, I), there is an integer $u$, such that

(10) \[ u = 1 \pmod{4}, \]

(11) \[ (u, 4p_1, \ldots, p_r) = 1, \]

(12) \[ p = u \pmod{4p_1, \ldots, p_r} \]

implies

\[ \left( \frac{- (\alpha - \beta)^2}{p} \right) = -1 \]

for any prime $p$.

Consider the pair of congruences

(13) \[ \begin{cases} x = u \pmod{4p_1, \ldots, p_r} \\ x = 1 \pmod{4\alpha \beta} \end{cases} \]
From the identity

\[(\alpha + \beta)^2 - 4\alpha \beta = (\alpha - \beta)^2,\]

and from the assumption

\[(\alpha + \beta, \alpha \beta) = 1,\]

it follows

\[1 = (\alpha \beta, p_1, \cdots, p_r) \leq (\alpha \beta, \pm \alpha^2 p_1, \cdots, p_r) = (\alpha \beta, (\alpha - \beta)^2) = 1.\]

Hence

\[(4p_1, \cdots, p_r, 4\alpha \beta) = 4(p_1, \cdots, p_r, \alpha \beta) = 4.\]

But, by (10), \(4|u - 1\), hence (13) has a solution \(u', \) i.e.,

\[(14) \quad u' = u (\text{mod } 4p_1, \cdots, p_r), \quad u' = 1 (\text{mod } 4\alpha \beta).\]

Let \(p\) be a prime satisfying \(p = 1 (\text{mod } 4\alpha \beta)\). If \(\alpha \beta\) is odd, then, according to the properties of the Jacobi symbol

\[\left( \frac{\alpha \beta}{p} \right) = \left( \frac{p}{\alpha \beta} \right) = \left( \frac{1}{\alpha \beta} \right) = 1.\]

If \(\alpha \beta\) is even, then \(p = 1 (\text{mod } 8)\), and \(\alpha \beta = 2^k t, \) where \(k \geq 1\) and \(2 \nmid t.\)

Then

\[\left( \frac{\alpha \beta}{p} \right) = \left( \frac{2^k}{p} \right) \left( \frac{t}{p} \right) = \left( \frac{p}{t} \right) = \left( \frac{1}{t} \right) = 1.\]

in both cases

\[\left( \frac{\alpha \beta}{p} \right) = 1.\]
Combining the last result with (11), (12) and (14), we conclude

(15) \[ (u', 4p_1, \ldots, p_r) = 1, \]

(16) If \( p = u' \pmod{4p_1, \ldots, p_r} \), then

\[ \left( \frac{- (\alpha - \beta)^2}{p} \right) = \left( \frac{- \alpha \beta (\alpha - \beta)^2}{p} \right) = -1, \]

for any prime \( p \).

We shall now show that if

(17) \[ p = u' \pmod{4p_1, \ldots, p_r}, \]

then \( p \mid D' \). Indeed, if \( p \mid D' \), then, by (1), \( p \mid \sqrt{n} \), hence \( p \mid \sqrt{s^2} \). Hence, by (9),

\[ p \mid (\alpha - \beta)^2 \beta^2 + 4 (\alpha \beta)^n. \]

Putting in Lemma 3:

\[ x = D', \quad y = 2, \quad a = (\alpha - \beta)^2, \quad b = (\alpha \beta)^n, \]

we have

\[ \left( \frac{- (\alpha \beta)^n (\alpha - \beta)^2}{p} \right) = 1. \]

If \( n \) is even, then

\[ 1 = \left( \frac{- (\alpha \beta)^n (\alpha - \beta)^2}{p} \right) = \left( \frac{- (\alpha - \beta)^2}{p} \right), \]

If \( n \) is odd, then
\[
1 = \left( \frac{-(\alpha \beta)^n (\alpha - \beta)^2}{p} \right) = \left( \frac{-\alpha \beta (\alpha - \beta)^2}{p} \right).
\]

Both cases contradict (16). The theorem now follows from (17), (15), and Lemma 1, II.

II) Suppose \((\alpha - \beta)^2 = m^2\), where \(m\) is an integer and \(\alpha \beta \neq 0\). Then (9) becomes

\[
S_n^2 = (mD_n)^2 + 4(\alpha \beta)^n.
\]

This formula implies, by Lemma 3, if

\[
p | D_n
\]

(and hence \(p | S_n^2\)), then

\[
\left( \frac{-(\alpha \beta)^n}{p} \right) = 1,
\]

for any odd prime \(p\). Consider now the three following cases.

**Case 1**: \(\alpha \beta = n^2 \cdot 2^k\), where \(k \geq 0\). Then, if \(p = -1 \pmod{8}\), then

\[
\left( \frac{-(\alpha \beta)^n}{p} \right) = \left( \frac{-1}{p} \right) \left( \frac{2}{p} \right)^k = -1,
\]

and hence, by (19), \(p \nmid D_n\).

**Case 2**: \(\alpha \beta = n^2 \cdot 2^k \cdot q_1, \ldots, q_r\), where \(k \geq 0\), \(r \geq 1\), \(q_1, \ldots, q_r\) are distinct odd primes, and \(t = q_1 \cdot \ldots \cdot q_r = 1 \pmod{4}\).

Consider the pair of congruences

\[
\begin{cases}
x = -1 \pmod{8} \\
x = 1 \pmod{t}
\end{cases}
\]

Since \((t, 8) = 1\), (20) has a solution \(u\). This solution satisfies

\[
(u, 8t) = 1.
\]
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If $p - u \pmod{8t}$ is a prime, then

$$\left( \frac{-\varphi(x)}{p} \right)^n = \left( \frac{-1}{p} \right) \left( \frac{2}{p} \right)^{kn} \left( \frac{t}{p} \right)^n = -\left( \frac{1}{t} \right) = -1,$$

and hence, by (19), $p \not\mid D_{8n}$.

**Case 3:** Everything as in Case 2, except that $t = -1 \pmod{4}$.

Choose a quadratic nonresidue $c$ modulo $q_1$, i.e.,

$$\left( \frac{c}{q_1} \right) = -1.$$ 

Consider the system of congruences

$$\begin{cases} 
  x = -1 \pmod{8} \\
  x = c \pmod{q_1} \\
  x = 1 \pmod{q_2} \\
  \vdots \\
  x = 1 \pmod{q_r} 
\end{cases}$$

If $r = 2$, or the system

$$\begin{cases} 
  x = -1 \pmod{8} \\
  x = c \pmod{q_1} 
\end{cases}$$

if $r = 1$. Since $q_1, \ldots, q_r$ are distinct odd primes, (23) and (24) have a solution $v$. $v$ satisfies:

$$\left( v, 8t \right) = 1,$$

If $p = v \pmod{8t}$ is a prime, then

$$\left( \frac{-\varphi(x)}{p} \right)^n = \left( \frac{-1}{p} \right) \left( \frac{2}{p} \right)^{kn} \left( \frac{t}{p} \right)^n = (-1)(+1) \left[ -\left( \frac{p}{t} \right) \right]^n.$$ 

$$= -\left( -\left( \frac{p}{q_1} \right) \left( \frac{p}{q_2} \right) \cdots \left( \frac{p}{q_r} \right) \right)^n$$

$$= -\left( -\left( \frac{c}{q_1} \right) \left( \frac{1}{q_2} \right) \cdots \left( \frac{1}{q_r} \right) \right)^n = 1.$$ 

[Cont. on p. 406.]