ANY LUCAS NUMBER \( L_{5p} \), FOR ANY PRIME \( p \geq 5 \), HAS AT LEAST TWO DISTINCT PRIMITIVE PRIME DIVISORS

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**Proof.** It is well known that, for any positive integer \( n \), \( L_{5n}/L_n = A_nB_n \), where

\[
A_n = 5F_n^2 - 5F_n + 1, \quad B_n = 5F_n^2 + 5F_n + 1, \quad A_n < B_n, \quad (A_n, B_n) = 1,
\]

where \( F_n \) denotes a Fibonacci number (compare, e.g., Recurring Sequences, Jerusalem, 1966, pp. 16-21. For \( n = 5 \) we have: \( A_n = 101, B_n = 151 \), and the statement is true. In order to prove it for \( p > 5 \), it is sufficient to show that the greatest non-primitive divisor of \( L_{5p}, p > 5 \), is smaller than \( A_p \), hence the greatest primitive divisor of \( L_{5p} \) is greater than \( B_p \), hence both \( A_n \) and \( B_n \) have primitive divisors, and since \( (A_n, B_n) = 1 \), \( A_n \) has a primitive prime divisor \( a \), \( B_n \) has a primitive prime divisor \( b \), and \( a \neq b \).

Now, the greatest non-primitive divisor of \( L_{5p} \) is \( L_{5}L_{p} = 11L_{p} \), and we have to show that \( 11L_{p} < A_p \) for any prime \( p > 5 \). We shall show that \( 11L_{n} < A_n \) for any positive integer \( n > 5 \). The proof is based on the following two inequalities:

\begin{align*}
(1) \quad & L_n < 3F_n \quad (n > 2), \\
(2) \quad & 33 < 5(F_n - 1) \quad (n > 5).
\end{align*}

Equation (1) is easily verified for \( n = 3, 4 \). If (1) is valid for \( n, n + 1 \), its validity for \( n + 2 \) follows by addition of the corresponding inequalities side-wise. Similarly (2) is shown. Hence

\[
11L_n < 11 \cdot 3F_n = 33F_n < 5(F_n - 1)F_n = 5F_n^2 - F_n < 5F_n^2 - F_n + 1 = A_n.
\]

This completes the proof.

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