ANY LUCAS NUMBER L_{5p}, FOR ANY PRIME $p \geq 5$, HAS AT LEAST TWO DISTINCT PRIMITIVE PRIME DIVISORS

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Proof. It is well known that, for any positive integer n, $L_{5n}/L_n = A_n B_n$, where

$$A_n = 5F_n^2 - 5F_n + 1, \quad B_n = 5F_n^2 + 5F_n + 1, \quad A_n < B_n, \quad (A_n, B_n) = 1,$$

where F_n denotes a Fibonacci number (compare, e.g., Recurring Sequences, Jerusalem, 1966, pp. 16-21. For $n = 5$ we have: $A_n = 101$, $B_n = 151$, and the statement is true. In order to prove it for $p > 5$, it is sufficient to show that the greatest non-primitive divisor of L_{5p}, $p > 5$, is smaller than A_p, hence the greatest primitive divisor of L_{5p} is greater than B_p, hence both A_n and B_n have primitive divisors, and since $(A_n, B_n) = 1$, A_n has a primitive prime divisor a, B_n has a primitive prime divisor b, and a $\neq$ b.

Now, the greatest non-primitive divisor of L_{5p} is $L_5 L_p = 11 L_p$, and we have to show that $11 L_p < A_p$ for any prime $p > 5$. We shall show that $11 L_n < A_n$ for any positive integer $n > 5$. The proof is based on the following two inequalities:

$$(1) \qquad L_n < 3F_n \quad (n > 2),$$

$$(2) \qquad 33 < 5(F_n - 1) \quad (n > 5).$$

Equation (1) is easily verified for $n = 3, 4$. If (1) is valid for $n, n+1$, its validity for $n+2$ follows by addition of the corresponding inequalities sidewise. Similarly (2) is shown. Hence

$$11 L_n < 11 \cdot 3F_n = 33 F_n < 5(F_n - 1)F_n = 5F_n^2 - F_n < 5F_n^2 - F_n + 1 = A_n.$$

This completes the proof.

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